Intersection cohomology with torus actions of complexity one and intersection space complexes

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Success is not final, failure is not fatal;
it is the courage to continue that counts.
Winston Churchill

Lo importante es el camino y en él,
caer, levantarse, insistir, aprender.
Mago de Oz

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Contents

Chapter 1

Introduction / Introducción

ENGLISH

The purpose of this thesis is studying different homological theories which produce Poincaré duality on singular spaces. We can divide our results in two different parts. The first one is an analysis of the intersection cohomology of varieties endowed with an effective action of an algebraic torus with complexity 1. In the second part, we generalize the definition of intersection space given by Banagl and we define a class of cohomologically constructible complexes of sheaves which recovers the cohomology of the intersection space if it exists. Now, we introduce both parts separately.

1.1 Intersection cohomology with torus actions of complexity one.

This part of the thesis is a joint work with Kevin Langlois and most of the results are included in our paper [AL17].

Let $\mathbb{T}$ be an algebraic torus. Let us recall that a $\mathbb{T}$-*variety* is a normal variety endowed with an effective $\mathbb{T}$-action. The *complexity* of a $\mathbb{T}$-variety X is the non-negative number $\dim(X) - \dim(\mathbb{T})$; it corresponds to the transcendence degree over $\mathbb{C}$ of the field extension $\mathbb{C}(X)^{\mathbb{T}}$ of invariant rational functions on X. By a result of Rosenlicht (see [Ros63]), the complexity of X is the codimension of a general $\mathbb{T}$-orbit. Moreover, if the $\mathbb{T}$-variety X is of complexity one, then the inclusion

$$\mathbb{C}(X)^{\mathbb{T}} \subseteq \mathbb{C}(X)$$

yields a map

$$\pi : X \dashrightarrow Y$$

to a smooth projective curve Y. Here Y is the projective curve obtained from the algebraic function field $\mathbb{C}(X)^{\mathbb{T}}$ of one variable. The map π is called the *rational quotient* of X. We say that the complexity-one $\mathbb{T}$-variety X is *contraction-free* if the rational quotient map given by the $\mathbb{T}$-action on X is a regular morphism.

The best concrete examples of $\mathbb{T}$-varieties are *toric varieties* (corresponding to complexity zero) and affine $\mathbb{C}^*$-surfaces. Both admit a combinatorial description. The first one can be defined by a fan of strongly convex polyhedral cones in the rational vector space $N_{\mathbb{Q}}$ associated with the lattice N of one-parameter subgroups of the torus $\mathbb{T}$ (see for instance [Ful93, CLS11]). The second one can be described by a $\mathbb{Q}$-divisor or a pair of $\mathbb{Q}$-divisors on a smooth algebraic curve (see [FZ03] for details). Note that contraction-free $\mathbb{T}$-varieties of complexity one were studied by Mumford in [KKMS73, Chapter IV]. These combinatorial descriptions admit a generalization to the setting of $\mathbb{T}$-varieties (see [AH06, AHS08, Tim08, Lan14]). The description in [AH06] of an affine $\mathbb{T}$-variety is in term of a divisor on a normal variety where its coefficients are polyhedra in $N_{\mathbb{Q}}$. Such a combinatorial object is called a *polyhedral divisor*. More generally, the combinatorial description introduced in [AHS08] for a $\mathbb{T}$-variety involves a *divisorial fan* which corresponds to a finite set of polyhedral divisors (with some additional conditions).

One of the results of this thesis is an explicit description (in terms of divisorial fans) of the intersection cohomology Betti numbers of projective contraction-free $\mathbb{T}$-varieties of complexity one (see Theorem 3.2.8). As an intermediate step, we compute the classical Betti numbers for every smooth projective $\mathbb{T}$-variety of complexity one (see Propositions 3.1.4 and 3.1.7). Proposition 3.1.7 was also obtained independently in [LLM16, Section 2]. These results can be related to classical ones in the field of intersection cohomology with a torus action. See [Kir88] for a general description in the projective case using the Bialynicki-Birula decomposition. See also [Sta87, DL91, Fie91, BBKK99] for the toric projective case which is related to h-polynomials and [FK86] for the case of affine $\mathbb{C}^*$-surfaces.

To motivate our result, let us make some comments on the description in [Sta87] for the toric projective case. In the sequel, for every algebraic variety V we will denote by $P_V(t) \in \mathbb{Z}[t]$ the Poincaré polynomial of V which is the generating function of the intersection cohomology Betti numbers of V (see 2.3.9 and 2.3.10). Let X be a projective toric variety (for the torus $\mathbb{T}$) with defining fan Σ_X. Then, since X is projective, the fan Σ_X is the normal fan of a rational polytope Q. In particular, the set of faces of dimension i of Q is in bijection with the set of cones of codimension i of Σ_X. In [Sta87] a polynomial $h(\Lambda; t)$ (called *h-polynomial*) depending on a polytope Λ is introduced, so that we have the equality $P_X(t) = h(Q; t^2)$ (see [Sta87, Theorem 3.1]). In the smooth case, the polynomial $h(Q; t^2)$ can be defined by the relation

$$h(Q; t^2) = \sum_{i=0}^{n} f_i(Q)(t^2 - 1)^i,$$

where $f_i(Q)$ is the number of faces of dimension i. However, in the non-smooth case, the polynomial above on the right-hand side has generally negative coefficients and therefore the definition of an h-polynomial is different (see [Sta87, Section 2] and also the remainder in the end of Section 2.3.1). Our main result is an adaptation of the description of the h-polynomials explained above (see [Sta87, Theorem 3.1]) to the setting of torus actions of complexity one.

Let us introduce some notation in order to explain our result. Let $\mathscr{E}$ be a divisorial fan on a smooth projective curve Y corresponding to a singular projective contraction-free $\mathbb{T}$-variety $X(\mathscr{E})$ of complexity one. We recall the definition of the divisorial fan $\mathscr{E}$ and the construction of the variety $X(\mathscr{E})$ in 2.4. Note that the curve Y is the quotient of the $\mathbb{T}$-action on $X(\mathscr{E})$. Let us denote by $\mathrm{supp}(\mathscr{E})$ the support of $\mathscr{E}$ which corresponds to points $y \in Y$ where the fibre of the quotient map is non-trivial (see Section 2.4 for a precise definition). Then, similarly to the toric case, one can attach rational polytopes $Q(\mathscr{E})$ and $Q_y(\mathscr{E})$ for every $y \in \mathrm{supp}(\mathscr{E})$ (see 3.1 for the construction of these objects). Our result can be stated as follows.

Theorem 1.1.1. *Let g be the genus of the curve Y and let r be the cardinality of the finite set* $\mathrm{supp}(\mathscr{E})$*. Then we have the equality*

$$P_{X(\mathscr{E})}(t) = ((1-r)t^2 + 2gt + 1 - r)h(Q(\mathscr{E}); t^2) + \sum_{y \in \mathrm{supp}(\mathscr{E})} h(Q_y(\mathscr{E}); t^2).$$

In particular, one can see that when $X(\mathscr{E})$ is a rational variety, the odd rational intersection cohomology groups of $X(\mathscr{E})$ vanish as in the toric projective case.

To obtain this result, we adapt to our setting a version of the decomposition theorem in [CMM15] given for toric fibrations (see Proposition 3.2.4). This allows us to show the result by induction on the dimension of $X(\mathscr{E})$ starting with a projective desingularization of $X(\mathscr{E})$ given by a subdivision of divisorial fans (see 3.2.2 for the definition of a subdivision of divisorial fans).

Our version of the decomposition theorem is expressed in terms of families of natural numbers called *s-sequences* which are related to the topology of toric proper maps. The subvarieties appearing in the decomposition theorem are all $\mathbb{T}$-stable. They are parametrized by a set $SH(\mathscr{E})$ depending combinatorially on the divisorial fan $\mathscr{E}$. For an algebraic variety X we will denote by IC_X its intersection cohomology complex with rational coefficients and middle perversity. Our result (see Proposition 3.2.4) can be enunciated as follows.

Theorem 1.1.2. *Let $\mathscr{E}$ be a divisorial fan on (Y, N) corresponding to a contraction-free $\mathbb{T}$-variety $X(\mathscr{E})$ and let $\mathscr{E}'$ be a subdivision of the divisorial fan $\mathscr{E}$. Consider the birational proper equivariant morphism*

$$f : X(\mathscr{E}') \to X(\mathscr{E})$$

given by the subdivision $\mathscr{E}'$. Then we have an isomorphism of perverse sheaves

$$f_\star IC_{X(\mathscr{E}')} \simeq \bigoplus_{\tau \in SH(\mathscr{E})} \bigoplus_{b \in \mathbb{Z}} (i_\tau)_\star IC_{V(\tau)}^{\oplus s_{\tau,b}}[-b]$$

in the derived category $D^b(X(\mathscr{E}); \mathbb{Q})$, where $s_{\tau,b}$ is an s-sequence of the subdivision $\mathscr{E}'$ and $i_\tau : V(\tau) \to X(\mathscr{E})$ is the inclusion.

Another way to determine $P_{X(\mathscr{E})}(t)$ would be to consider the *stratified multiplicative property* for intersection homology introduced in [CMS08]. This method involves a complex algebraic Whitney stratification on the base curve Y so that the quotient map $\pi : X(\mathscr{E}) \to Y$ becomes a stratified submersion. Moreover, we emphasize that the h-polynomial $h(Q_y(\mathscr{E}); t^2)$ is not equal to the Poincaré polynomial of the special fibre $\pi^{-1}(y)_{\mathrm{red}}$, but to the Poincaré polynomial of a certain toric variety of dimension $\dim X(\mathscr{E})$. This latter appearing in our induction process allows us to have a simple expression of $P_{X(\mathscr{E})}(t)$. Thus using [CMS08] a substantial work is needed in order to recover Theorem 1.3.1.

Our results aim to give a better understanding of the intersection cohomology for complexity-one torus actions on projective normal varieties. Indeed, one can construct every such a $\mathbb{T}$-variety from a contraction-free one by contracting certain families of orbits (compare with [AH06, Theorem 3.1 (ii)]). Hence assuming that these contractions are projective, a natural problem would be to describe the decomposition theorem for such maps. Thus using Theorem 1.3.1 this would give a description of the intersection cohomology Betti numbers in this setting.

1.2 Intersection spaces and cohomologically constructible complexes associated to them

This part of the thesis is a joint work with Javier Fernández de Bobadilla.

Intersection spaces have been recently introduced by Banagl as a Poincaré duality homology theory for topological pseudomanifolds which is an alternative to Goreski and McPherson intersection homology. When they are available they present the advantages of being spatial modifications of the given topological pseudomanifold, to which one can later apply algebraic topology functors in order to obtain invariants. In this sense, if one applies (reduced) singular cohomology one obtains a homology theory with internal cup products and a Poincaré duality is satisfied between the homology theories corresponding to complementary perversities. Moreover, one can apply many other functors leading to richer invariants. The idea of intersection spaces was sketched for the first time in [Ban09] and was fully developed for spaces with isolated singularities in [Ban10].

In [Ban10] Banagl carefully analyzed the case of quintic 3-folds with ordinary double points appearing in the conifold transition and noticed that, in the same way that intersection cohomology gives the cohomology of a small resolution, cohomology of intersection spaces gives the cohomology of a smoothing in this case. This fitted with predictions motivated by string theory (see Banagl papers for full explanations).

This motivated further work by Banagl, Maxim and Budur ([BM1], [BM2], [BBM], [Max]) in which the relation between the cohomology of intersection spaces for the middle perversity and the Milnor fibre of a hypersurface X with isolated singularities is analyzed. The latest evolution of the results of these papers, contained in [BBM], is the construction of a perverse sheaf in X whose hypercohomology computes the cohomology of the intersection space of X in all degrees except for the top degree.

Such a perverse sheaf is a modification of the nearby cycle complex and, in fact, when the monodromy is semi-simple in the eigenvalue 1, the middle perversity intersection space perverse sheaf is a direct summand of the nearby cycle complex.

The results described up to now are valid for only isolated singularities (sometimes even assuming that they are hypersurface singularities). In [Ban10] Banagl generalizes the construction for the case of topological pseudomanifolds with two strata and trivial link fibration and sketches a method for more general class of non-isolated singularities. In [BaCh] intersection spaces are constructed for the case of two strata assuming non-trivial conditions on the fibration by links. The only case in which intersection spaces are constructed for a topological pseudomanifold with more than two strata is in [Ban12]. There, the depth 1 strata are circles or intervals and the depth 2 strata are isolated singularities; in this case, it is the topology of the strata which is very restrictive.

In [BM1] the following open questions are proposed: Is there a sheaf theoretic approach to intersection spaces, similar to the one of Goreski and McPherson [GM83] for Intersection Cohomology?. Up to which kind of singularities the intersection space constructions can be extended? Is intersection space cohomology of algebraic varieties endowed with a Hodge structure?

The papers [BM2], [BBM] are contributions to the first and third question for the case of isolated singularities. The paper [BaHu] is also a contribution towards the third question.

This thesis is a contribution to the first and the second questions questions formulated above for the general singularity case. First, we prove that if a Banagl intersection space exists for a topological pseudomanifold and a given perversity, then there exist a cohomologically constructible complex of sheaves in our original space X that satisfies a set of properties of the same kind that those that characterize intersection cohomology complexes in [GM83]; we call this complex an intersection space complex for the given perversity. Its hypercohomology recovers the reduced cohomology of the intersection space in the case of isolated singularities. In the case of depth 1 topological pseudomanifold, it recovers the cohomology of the intersection space relative to the singular stratum (like in [Ban10]), which is the one that satisfies Poincaré duality for complementary perversities. For depth 2 and higher, if the dimension of the strata is sufficiently high, the intersection space construction is intrinsically a construction of pairs of spaces, as we will see below; the hypercohomology of our intersection space complex computes the rational cohomology of the pair of spaces. Moreover, in this thesis we construct the Banagl intersection space for any topological pseudomanifold with trivial link fibrations such that the associated trivializations verify some compatibility conditions.

Next, we leave the realm of topology and shift to a sheaf theoretic viewpoint studying under which conditions intersection space complexes exist. We find obstructions for existence and uniqueness of intersection space complexes and give spaces parametrizing the possible intersection space complexes in case that the obstruction for existence vanish. Both of these obstructions vanish in the case of isolated singularities and the obstruction for existence also vanish in the case considered in [Ban12], as one should expect.

We turn to analyze classes of topological pseudomanifolds in which we can prove the existence of intersection space complexes. We show that they exist for any perversity when the successive link fibrations are trivial. This includes the case of toric varieties. We also prove the existence if the homological dimension of the strata with respect to local systems is at most 1. This includes the case treated in [Ban12]. On the other hand, building on the obstructions for existence, we show that, if the intersection space complex exist, then certain differentials on the local to global spectral sequence for the link fibration have to vanish. Using this, we produce the first examples of topological pseudomanifolds such that intersection space complexes do not exist for given perversities. As a consequence, Banagl intersection spaces can not exist either. One of the examples is a normal algebraic variety whose stratification has depth 1 and whose transversal singularity is an ordinary double point of dimension 3 (those appearing in the conifold transition examples); the perversity used is the middle one.

Finally, we turn to duality questions. We show that the Verdier dual of an intersection space complex with a given perversity is an intersection space complex with the complementary perversity. The proof resembles the one given in [GM83] for intersection cohomology complexes. However, since (unlike intersection cohomology complexes) intersection space complexes are not unique, this does not yield self dual sheaves for the middle perversity on algebraic varieties. In the case of depth 1 stratifications, we prove that generic choices of the intersection space complex yield the same Betti numbers in hypercohomology and we obtain Poincaré duality at the level of generic Betti numbers for complementary perversities.

In a recent paper [Ge], Genske takes a new viewpoint: instead of only giving up the topological construction and focusing in producing a complex of sheaves at the original space, he constructs a complex of vector spaces which is related with the complex computing the (co)homology of the original space X, but that satisfies Poincaré duality. The construction is valid for any analytic variety (Poincaré duality is satisfied in the compact case). His construction is a bit further to original Banagl ideas than ours, since his procedure is to make a modification which is global in a neighbourhood of the singular set, instead of stratifying it conveniently and making a fibrewise construction.

Finally, in order to finish our review of existing results, let us mention the rational Poincaré spaces approach developed in [Kl].

ESPAÑOL

El propósito de esta tesis es estudiar diferentes teorías homológicas que producen dualidad de Poincaré en espacios singulares. Podemos dividir nuestros resultados en dos partes diferentes. La primera es un análisis de la cohomología de intersección de variedades con una acción efectiva de un toro algebraico con complejidad 1. En la segunda parte, generalizamos la definición de espacio de intersección dada por Banagl y definimos una clase de complejos de haces cohomológicamente constructibles que recupera la cohomología del espacio de intersección en caso de que este exista. A continuación, introducimos ambas partes por separado.

1.3 Cohomología de intersección con acciones del toro de complejidad uno.

Esta parte de la tesis es un trabajo conjunto con Kevin Langlois y la mayor parte de los resultados están incluidos en nuestro artículo [AL17].

Sea $\mathbb{T}$ un toro algebraico. Recordemos que una $\mathbb{T}$-*variedad* es una variedad normal dotada con una $\mathbb{T}$-acción efectiva. La *complejidad* de una $\mathbb{T}$-variedad X es el número no negativo $\dim(X) - \dim(\mathbb{T})$; este se corresponde con el grado de transcendencia sobre $\mathbb{C}$ de la extension $\mathbb{C}(X)^{\mathbb{T}}$ de funciones racionales en X invariantes por la acción del toro. Por un resultado de Rosenlicht (vease [Ros63]), la complejidad de X es la codimensión de una $\mathbb{T}$-órbita general. Además, si la $\mathbb{T}$-variedad es de complejidad 1, la inclusión

$$\mathbb{C}(X)^{\mathbb{T}} \subseteq \mathbb{C}(X)$$

induce una aplicación

$$\pi : X \dashrightarrow Y$$

a una curva proyectiva lisa Y. Aquí Y es la curva proyectiva cuyo cuerpo de funciones es $\mathbb{C}(X)^{\mathbb{T}}$. La aplicación π es el *cociente racional* de X. Decimos que la $\mathbb{T}$-variedad de complejidad 1 X es *libre de contración* si el cociente racional dado por la $\mathbb{T}$-acción en X es un morfismo regular.

Los mejores ejemplos de $\mathbb{T}$-variedades son las *variedades tóricas* (que se corresponden a las variedades de complejidad 0) y las $\mathbb{C}^*$-superficies afines. Ambas admiten una descripción combinatoria. Las primeras pueden ser definidas por un abanico the conos poliedrales fuertemente convexos en el espacio vectorial racional $N_{\mathbb{Q}}$ asociado con el retículo N de los subgrupos uniparamétricos del toro $\mathbb{T}$ (vease por ejemplo [Ful93, CLS11]). Las segundas pueden ser descritas por un $\mathbb{Q}$-divisor o un par de $\mathbb{Q}$-divisores en una curva algebraica lisa (vease [FZ03] para los detalles). Notese que las $\mathbb{T}$-variedades libres de contracción de complejidad uno fueron estudiadas por Mumford en [KKMS73, Chapter IV]. Estas descripciones combinatorias admiten una generalización al marco de las $\mathbb{T}$-variedades (vease [AH06, AHS08, Tim08, Lan14]). La descripción en [AH06] de una $\mathbb{T}$-variedad afín se realiza en terminos de un divisor en una variedad normal cuyos coeficientes son poliedros en $N_{\mathbb{Q}}$. Este objeto combinatorio se denomina *divisor poliedral.* De modo más general, la descripción combinatoria introducida en [AHS08] para una $\mathbb{T}$-variedad envuelve a un *abanico de divisores* lo que se corresponde con un conjunto finito de divisores poliedrales (con algunas condiciones adicionales).

Uno de los resultados de esta tesis es una descripción explícita (en términos de los abanicos de divisores) de los números de Betti de la cohomología de intersección de las $\mathbb{T}$-variedades libres de contracción de complejidad uno (vease teorema 3.2.8). Como un paso intermedio, calculamos los números de Betti clásicos para cualquier $\mathbb{T}$-variedad proyectiva de complejidad uno (veanse las Proposiciones 3.1.4 y 3.1.7). La Proposición

3.1.7 fue también obtenida de forma independiente en [LLM16, Section 2]. Estos resultados se pueden relacionar con los resultados clásicos en el campo de la cohomología de intersección con una acción de un toro. El lector puede ver [Kir88] para una descripción general en el caso proyectivo usando la descomposición de Bialynicki-Birula. Consultense también [Sta87, DL91, Fie91, BBKK99] para ver el caso de variedades tóricas proyectivas, que se relacionan con los h-polinomios, y [FK86] para el caso de $\mathbb{C}^*$-superficies.

Para motivar nuestro resultado, hagamos algunos comentarios de la descripción en [Sta87] para el caso de variedades tóricas proyectivas. A partir de ahora, para toda variedad algebraica V denotaremos con $P_V(t) \in \mathbb{Z}[t]$ al polinomio de Poincaré de V, que es la función generadora de los números de Betti de la cohomología de intersección de V (ver 2.3.9 y 2.3.10). Sea X una variedad tórica proyectiva (para el toro $\mathbb{T}$) cuyo abanico asociado es Σ_X. Entonces, como X es proyectiva, el abanico Σ_X es el abanico normal asociado a un politopo racional Q. En particular, el conjunto de caras de dimensión i de Q está en biyección con el conjunto de conos de codimensión i de Σ_X. En [Sta87] se introduce un polinomio $h(\Lambda; t)$ (llamado *h-polinomio*) que depende de un politopo Λ, tal que tenemos la igualdad $P_X(t) = h(Q; t^2)$ (ver [Sta87, Theorem 3.1]). En el caso liso, el polinomio $h(Q; t^2)$ puede definirse con la relación

$$h(Q; t^2) = \sum_{i=0}^{n} f_i(Q)(t^2 - 1)^i,$$

donde $f_i(Q)$ es el número de caras de dimensión i. Sin embargo, en el caso no liso, el lado derecho de la igualdad anterior tiene, generalmente, coeficientes negativos y, por tanto, la definición de h-polinomio es diferente (ver [Sta87, Section 2] y también el recordatorio al final de la Sección 2.3.1). Nuestro resultado principal es una adaptación de la descripción del h-polinomio explicada antes (ver [Sta87, Theorem 3.1]) al marco de acciones del toro con complejidad uno.

Introduzcamos algo de notación para explicar nuestro resultado. Sea $\mathscr{E}$ un abanico de divisores en una curva proyectiva lisa Y asociado a una $\mathbb{T}$-variedad singular proyectiva libre de contracción $X(\mathscr{E})$ de complejidad uno. En 2.4 recordamos la definición de abanico de divisores $\mathscr{E}$ y la construcción de la variedad $X(\mathscr{E})$. Notese que la curva Y es el cociente de la $\mathbb{T}$-acción en $X(\mathscr{E})$. Denotemos con $\operatorname{supp}(\mathscr{E})$ el soporte de $\mathscr{E}$ que se corresponde con los puntos $y \in Y$ donde la fibra de la aplicación cociente no es trivial (ver Seccion 2.4 para una definición precisa). Entonces, de modo similar al caso tórico, uno puede definir politopos racionales $Q(\mathscr{E})$ y $Q_y(\mathscr{E})$ para todo $y \in \operatorname{supp}(\mathscr{E})$ (vease 3.1 para la construcción de estos objetos). Nuestro resultado es el siguiente.

Theorem 1.3.1. *Sea g el género de la curva Y y sea r la cardinalidad del conjunto finito* $\operatorname{supp}(\mathscr{E})$. *Entonces, tenemos la igualdad*

$$P_{X(\mathscr{E})}(t) = ((1-r)t^2 + 2gt + 1 - r)h(Q(\mathscr{E}); t^2) + \sum_{y \in \operatorname{supp}(\mathscr{E})} h(Q_y(\mathscr{E}); t^2).$$

En particular, uno puede ver que cuando $X(\mathscr{E})$ es una variedad racional, los grupos impares de cohomología de intersección racional de $X(\mathscr{E})$ son 0 como en el caso de variedades tóricas proyectivas.

Para obtener este resultado, adaptamos a nuestro ámbito una versión del teorema de descomposición dado en [CMM15] para fibraciones tóricas (vease Proposicion 3.2.4). Esto nos permite probar el resultado por inducción sobre la dimensión de $X(\mathscr{E})$ empezando con una desingularización de $X(\mathscr{E})$ dada por una subdivisión de abanicos de divisores (vease 3.2.2 para la definición de subdivisión de abanicos de divisores).

Nuestra versión del teorema de descomposición se expresa en términos de familias de números naturales llamadas *s-sucesiones* que están relacionadas con la topología de morfismos tóricos propios. Las subvariedades que aparecen en el teorema de descomposición son todas $\mathbb{T}$-estables y están parametrizadas por un conjunto $SH(\mathscr{E})$ que depende combinatoriamente del abanico de divisores $\mathscr{E}$. Dada una variedad algebraica X, denotamos con IC_X a su complejo de cohomología de intersección con coeficientes racionales y perversidad intermedia. Nuestro resultado (vease Proposición 3.2.4) es el siguiente.

Theorem 1.3.2. *Sea $\mathscr{E}$ un abanico de divisores en (Y, N) que se corresponde con una $\mathbb{T}$-variedad libre de contracción $X(\mathscr{E})$ y sea $\mathscr{E}'$ una subdivisión del abanico de divisores $\mathscr{E}$. Consideremos el morfismo birracional propio y equivariante*

$$f : X(\mathscr{E}') \to X(\mathscr{E})$$

dado por la subdivisión $\mathscr{E}'$. Entonces, tenemos un isomorfismo de haces perversos

$$f_\star IC_{X(\mathscr{E}')} \simeq \bigoplus_{\tau \in SH(\mathscr{E})} \bigoplus_{b \in \mathbb{Z}} (i_\tau)_\star IC_{V(\tau)}^{\oplus s_{\tau,b}}[-b]$$

en la categoría derivada $D^b(X(\mathscr{E}); \mathbb{Q})$, donde $s_{\tau,b}$ es una s-sucesión de la subdivisión $\mathscr{E}'$ e $i_\tau : V(\tau) \to X(\mathscr{E})$ es la inclusión.

Otro modo de determinar $P_{X(\mathscr{E})}(t)$ sería considerar la *propiedad multiplicativa estratificada* para la homología de intersección introducida en [CMS08]. Este método implicaría utilizar una stratificación de Whitney compleja en la curva Y tal que la aplicación cociente $\pi : X(\mathscr{E}) \to Y$ sea una submersión estratificada. Además, nosotros enfatizamos que el h-polynomio $h(Q_y(\mathscr{E}); t^2)$ no es igual al polinomio de Poincaré de la fibra especial $\pi^{-1}(y)_{\mathrm{red}}$, si no que es igual al polinomio de Poincaré de cierta variedad tórica de dimensión $\dim X(\mathscr{E})$. Este hecho junto con el proceso de inducción nos permite dar una expresión simple de $P_{X(\mathscr{E})}(t)$. Por tanto, usando [CMS08] se requiere un trabajo sustancial para obtener el Teorema 1.3.1.

Nuestros resultados pretenden mejorar la comprensión de la cohomología de intersección en el caso de variedades normales proyectivas con una acción de un toro con complejidad uno. De hecho, uno puede construir cualquier $\mathbb{T}$-variedad a partir de una $\mathbb{T}$-variedad libre de contracción contrayendo ciertas familias de órbitas (vease [AH06, Theorem 3.1 (ii)]). Por lo tanto, asumiendo que estas contraciones son proyectivas, un problema natural sería describir el teorema de descomposición para estos morfismos. Entonces, usando el Teorema 1.3.1 obtendríamos una descripción de los números de Betti de la cohomología de intersección en este ámbito.

1.4 Espacios de intersección y complejos cohomológicamente constructibles asociados

Esta parte de la tesis es un trabajo conjunto con Javier Fernández de Bobadilla.

Los espacios de intersección han sido introducidos recientemente por Banagl como una teoría de homología con dualidad de Poincaré para pseudovariedades topológicas que es una alternativa a la homología de intersección de Goreski y McPherson. Cuando estos espacios se pueden definir, presentan la ventaja de ser modificaciones espaciales de las pseudovariedades topológicas, a las que después uno puede aplicar funtores algebraico-topológicos para obtener invariantes. En este sentido, si uno aplica cohomología singular (reducida) a estos espacios, obtiene una teoría de homología con un producto cup interno y la dualidad de Poincaré se satisface entre las teorías de homología correspondientes a perversidades complementarias. Además, uno puede aplicar muchos otros funtores para obtener más invariantes. La idea de los espacios de intersección fue esbozada por primera vez en [Ban09] y fue completamente desarrollada para espacios con singularidades aisladas en [Ban10].

En [Ban10] Banagl realizó cuidadosamente el caso de 3-variedades quínticas con puntos dobles ordinarios en la transición del conifold y noto que, del mismo modo que la cohomología de intersección da la cohomología de una resolución, la cohomología del espacio de intersección da la cohomología de un smoothing. Esto encaja con las prediciones de la teoría de cuerdas (veanse los artículos de Banagl para las explicaciones completas).

Esto motivó trabajos posteriores desarrollados por Banagl, Maxim y Budur ([BM1], [BM2], [BBM], [Max]) en los cuales se analiza la relación entre la cohomología de los espacios de intersección para la perversidad media y la fibra de Milnor de una hipersuperficie X con singularidades aisladas. La última evolución de los resultados en estos artículos, contenida en [BBM], es la construcción de un haz perverso en X cuya hipercohomología calcula la cohomología del espacio de intersección de X en todos los grados excepto el grado máximo. Este haz perverso es una modificación del complejo de ciclos próximos y, de hecho, cuando la monodromía es semi-simple en el valor principal 1, el haz perverso del espacio de intersección para la perversidad media es un sumando directo del complejo de ciclos próximos.

Los resultados descritos hasta ahora son válidos solamente en el caso de singularidades aisladas (en algunos casos incluso se necesita que sean singularidades de hipersuperficies). En [Ban10] Banagl generaliza la construcción para el caso de pseudovariedades topológicas con dos estratos y una fibración de links trivial y esboza un método para clases más generales de singularidades no aisladas. En [BaCh] los espacios de intersección se construyen para el caso de dos estratos asumiendo condiciones no triviales en la fibración de links. El único caso en el que los espacios de intersección se construyen para pseudovariedades topológicas con mas de dos stratos es en [Ban12]. Aquí, los estratos de profundidad 1 son círculos e intervalos y los estratos de profundidad 2 son singularidades aisladas; en este caso, la topología de los estratos es muy restrictiva.

En [BM1] se proponen las siguientes preguntas abiertas: ¿Hay alguna aproximación a los espacios de intersección a través de la teoría de haces similar a la de Goreski y McPherson [GM83] para la cohomología de intersección? ¿Para qué tipo de singularidades se puede extender la construcción de los espacios de intersección? ¿La cohomología de los espacios de intersección tiene una estructura de Hodge?

Los artículos [BM2] y [BBM] son contribuciones a la primera y a la tercera pregunta para el caso de singularidades aisladas. El artículo [BaHu] también es una contribución a la tercera pregunta.

Esta tesis es una contribución a la primera y a la segunda pregunta para el caso de singularidades generales. En primer lugar, probamos que si existe un espacio de intersección para una pseudovariedad topológica y una perversidad dada, entonces existe un complejo de haces cohomológicamente constructible en nuestro espacio original X que satisface un conjunto de propiedades del mismo tipo que aquellas que caracterizan los complejos de cohomología de intersección en [GM83]; llamamos a este complejo el complejo del espacio de intersección para la perversidad dada. Su hipercohomología recupera la cohomología reducida del espacio de intersección en el caso de singularidades aisladas. En el caso de pseudovariedades topológicas con profundidad 1, recupera la cohomología del espacio de intersección relativa al estrato singular (como en [Ban10]), que es la cohomología que satisface la dualidad de Poincaré para perversidades complementarias. Para profundidad 2 y superior, si la dimensión de los estratos es suficientemente grande, la construcción del espacio de intersección es intrínsicamente una construción de pares de espacios, como veremos más adelante; la hipercohomología de nuestro complejo del espacio de intersección calcula la cohomología racional del par de espacios. Además, en esta tesis se construye el espacio de intersección de Banagl para cualquier pseudovariedad topológica con fibraciones de link triviales tales que las trivializaciones asociadas verifican ciertas condiciones de compatibilidad.

A continuacion, dejamos el reino de topología y tomamos el punto de vista de la teoría de haces para estudiar bajo que condiciones existen los complejos del espacio de intersección. Encontramos obstrucciones para la existencia y la unicidad del complejo del espacio de intersección y damos espacios que parametrizan los posibles complejos del espacio de intersección en caso de que la obstrucción para la existencia se anule. Ambas obstrucciones se anulan en el caso de singularidades aisladas y la obstrucción para la existencia también se anula en el caso considerado en [Ban12], como uno esperaría.

Seguimos analizando clases de pseudovariedades topológicas en las cuales podemos probar la existencia de los complejos del espacio de intersección. Probamos que existen para cualquier perversidad cuando las fibraciones de links son triviales. Esto incluye el caso de variedades tóricas. También probamos que existen si la dimensión homológica de los estratos con respecto a los sistemas locales es menor o igual que 1. Esto incluye el caso tratado en [Ban12]. Por otra parte, construyendo las obstrucciones para la existencia, probamos que si el complejo del espacio de intersección existe, entonces ciertas diferenciales en la sucesión espectral de local a global se anulan. Usando esto, producimos los primeros ejemplos de pseudovariedades topológicas tales que el complejo del espacio de intersección no existe para determinadas perversidades. En consecuencia, los espacios de intersección de Banagl tampoco existen. Uno de los ejemplos es una variedad algebraica normal cuya estratifcación tiene profundidad 1 y cuya singularidad transversal es un punto doble ordinario de dimensión 3 (esos que aparecen en los ejemplos de la transición del conifold); la perversidad usada es la intermedia.

Finalmente, estudiamos las cuestiones relativas a la dualidad. Probamos que el dual de Verdier de un complejo del espacio de intersección con una perversidad dada es un complejo del espacio de intersección con la perversidad complementaria. La demostración es similar a la dada en [GM83] para los complejos de cohomología de intersección. Sin embargo, como (a diferencia de los complejos de cohomología de intersección) los complejos del espacio de intersección no son unicos, esto no implica la autodualidad de los complejos del espacio de intersección con la perversidad intermedia en variedades algebraicas. En el caso de estratificaciones con profundidad 1, probamos que elecciones genéricas de los complejos del espacio de intersección producen los mismos números de Betti en hipercohomología y obtenemos dualidad de Poincaré al nivel de los números de Betti genéricos para perversidades complementarias.

En un artículo reciente [Ge], Genske toma un nuevo punto de vista; en lugar de abandonar la construcción topológica y producir un complejo de haces en el espacio original, construye un complejo de espacios vectoriales que se relaciona con el complejo que calcula la (co)homología del espacio original X, pero satisface la dualidad de Poincaré. La construcción es válida para cualquier variedad analítica (la dualidad de Poincaré se satisface en el caso compacto). Su construcción esta un poco más lejos de la idea original de Banagl que la nuestra, ya que su procedimiento es hacer una modificación global en un entorno del conjunto singular, en lugar de estratificarlo y realizar una construcción relativa a las fibras.

Por último, para terminar nuestro repaso a los resultados previos, debemos mencionar la aproximación de espacios de Poincaré racionales desarrollada en [Kl].

Chapter 2

Preliminaries

In this chapter, we explain some previous notions about the derived category, spectral sequences, intersection cohomology and $\mathbb{T}$-varieties which we use along the thesis.

2.1 Preliminaries on Derived Category

In this section, we define the derived category and the right derived functors for any abelian category with enough injectives. Our description of these objects is the description in [Ban07, Chapter 2].

Although the construction is carried for any abelian category $\mathcal{A}$ with enough injectives, for the purpose of this thesis the reader can think of $\mathcal{A}$ as the category of sheaves of $\mathbb{Q}$-vector spaces over a topological space.

First, we define the *homotopic category* of $\mathcal{A}$, $\mathcal{K}(\mathcal{A})$, as follows.

- The objects of this category are the complexes in $\mathcal{A}$.
- The morphisms of this category are the homotopic classes of morphisms of complexes.

Remember that given two morphisms of complexes

$$f, g : A^\bullet \to B^\bullet$$

f is homotopic to g if, for every integer n, there exist a morphism
$h^n : A^{n+1} \to B^n$ such that

$$d_B^n \circ h^n + h^{n+1} \circ d_A^{n+1} = g - f,$$

where d_A is the differential of the complex $A^\bullet$ and d_B is the differential of the complex $B^\bullet$.

If f is a morphism of complexes, we denote its homotopic class by $[f]$.

The category $\mathcal{K}(\mathcal{A})$ is triangulated in the sense of [Ban07, Section 2.2] which we explain now.

Let $\mathcal{K}$ be an additive category with an automorphism $T : \mathcal{K} \to \mathcal{K}$. The reader can suppose $\mathcal{K}$ is $\mathcal{K}(\mathcal{A})$ and the automorphism T is the shift of complexes $A^\bullet \to A^\bullet[1]$.

Definition 2.1.1. A *triangle* in $\mathcal{K}$ is a sextuple (X, Y, Z, u, v, w) such that X, Y, Z ore objects in $\mathcal{K}$ and $u : X \to Y$, $v : Y \to Z$ and $w : Z \to T(X)$ are morphisms.

We also denote triangles (X, Y, Z, u, v, w) by

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$$

$$X \to Y \to Z \xrightarrow{[1]}$$

and

$$\begin{tikzcd} X \arrow[rr] & & Y \arrow[dl] \\ & Z \arrow[ul, "{[1]}"] & \end{tikzcd}$$

where the [1] means the morphism goes to $T(X)$.

Definition 2.1.2. $\mathcal{K}$ is a *triangulated category* if there exist a collection of triangles $\{(X, Y, Z, u, v, w)\}$ called *the distinguished triangles* verifying the following properties.

- Any triangle isomorphic to a distinguished triangle is distinguished.
- For every object X, the triangle $(X, X, 0, Id_X, 0, 0)$ is distinguished.
- Every morphism $u : X \to Y$ can be embedded in a distinguished triangle (X, Y, Z, u, v, w).
- Given two distinguished triangles (X, Y, Z, u, v, w) and (X', Y', Z', u', v', w') and a commutative diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "\alpha"] & Y \arrow[d, "\beta"] \\ X' \arrow[r, "u'"] & Y' \end{tikzcd}$$

  there exist a morphism $\gamma : Z \to Z'$ completing the morphism between triangles, that is, making the following diagram commutative:

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "\alpha"] & Y \arrow[r, "v"] \arrow[d, "\beta"] & Z \arrow[r, "w"] \arrow[d, "\gamma"] & T(X) \arrow[d, "T(\alpha)"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & T(X') \end{tikzcd}$$

  The morphism $\gamma : Z \to Z'$ is not unique in general.

- **The octahedral axiom.** Given three distinguished triangles

$$\begin{tikzcd} X \arrow[rr, "u"] & & Y \arrow[dl] \\ & Z' \arrow[ul, "{[1]}"] & \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[rr, "v"] & & Z \arrow[dl] \\ & X' \arrow[ul, "{[1]}"] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[rr, "v \circ u"] & & Z \arrow[dl] \\ & Y' \arrow[ul, "{[1]}"] & \end{tikzcd},$$

there exist a distinguished triangle

$$\begin{tikzcd}
Z' \arrow[rr] & & Y' \arrow[dl] \\
& X' \arrow[ul, "{[1]}"] &
\end{tikzcd}$$

which completes the diagram

$$\begin{tikzcd}
X \arrow[rr, "u"] \arrow[rrrr, bend left, "u \circ v"] & & Y \arrow[rr, "v"] \arrow[dl] & & Z' \arrow[dl] \arrow[ddll, bend left] \\
& Z' \arrow[ul, "{[1]}"] \arrow[dr] & & X' \arrow[ul, "{[1]}"'] \arrow[ll, "{[1]}"] & \\
& & Y' \arrow[ur] \arrow[uull, bend left, "{[1]}"] & &
\end{tikzcd}$$

such that the induced diagrams

$$\begin{tikzcd}
Z' \arrow[rr] \arrow[dr, "{[1]}"] & & Y' \arrow[dl, "{[1]}"] \\
& X &
\end{tikzcd}
\qquad
\begin{tikzcd}
X' \arrow[rr, "{[1]}"] \arrow[dr, "{[1]}"] & & Y \arrow[dl] \\
& Z' &
\end{tikzcd}
\qquad
\begin{tikzcd}
Z \arrow[rr] \arrow[dr] & & X' \arrow[dl] \\
& Y' &
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, "u"] & Y \\
Y' \arrow[u, "{[1]}"] \arrow[r] & X' \arrow[u, "{[1]}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[r, "v"] \arrow[d] & Z \arrow[d] \\
Z' \arrow[r] & Y'
\end{tikzcd}$$

are commutative.

The homotopic category $\mathcal{K}(\mathcal{A})$ with the automorphism $A^\bullet \to A^\bullet[1]$ is a triangulated category. We construct the distinguished triangles as follows.

Definition 2.1.3. Given a morphism of complexes in $\mathcal{A}$, $f : A^\bullet \to B^\bullet$, the *cone* of f is the complex $cone(f)$ such that

$$(cone(f))^n := A^{n+1} \oplus B^n$$

and whose differential is

$$d_f^n := \begin{pmatrix} -d_A^{n+1} & 0 \\ f^{n+1} & d_B^n \end{pmatrix},$$

where d_A is the differential of the complex $A^\bullet$ and d_B is the differential of the complex $B^\bullet$.

For any morphism of complexes $f : A^\bullet \to B^\bullet$, we have a triangle in $\mathcal{K}(\mathcal{A})$

$$(A^\bullet, B^\bullet, cone(f), [f], [i_2], [p_1]),$$

where $i_2^n : B^n \to (cone(f))^n$ is the canonical inclusion and $p_1^n : (cone(f))^n \to A^{n+1}$ is the canonical projection.

A triangle in $\mathcal{K}(\mathcal{A})$ is distinguished if it is isomorphic to $(A^\bullet, B^\bullet, cone(f), [f], [i_2], [p_1])$ for some morphism f. The fact that this set of triangles verify the properties of Definition 2.1.2 is proven in [Ban07, Theorem 2.3.1].

Remark 2.1.4. Given an exact sequence of complexes

$$0 \to A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \to 0,$$

it induces a distinguished triangle

$$A^\bullet \xrightarrow{[f]} B^\bullet \xrightarrow{[g]} C^\bullet \xrightarrow{[1]}$$

in $\mathcal{K}(\mathcal{A})$ if and only if $C^\bullet$ is homotopically equivalent to $cone(f)$.

There is a quasi-isomorphism

$$\begin{pmatrix} 0 \\ g^n \end{pmatrix} : (cone(f))^n \to C^n$$

but this morphism is not always a homotopy equivalence.

So, we are interested in a category such that the quasi-isomorphisms are isomorphisms. This is the derived category. To introduce this category, we need the definitions of multiplicative system and localization of a category which appear in [Ban07, Section 2.4.1].

Definition 2.1.5. Let $\mathcal{K}$ be a category. A collection S of morphism of $\mathcal{K}$ is a *multiplicative system* if it verifies the following properties.

- For every object X of $\mathcal{K}$, Id_X is in S.
- The composition of two morphisms of S is also in S.
- For every morphism of $\mathcal{K}$, $u : X \to Y$, and every morphism in S, $s : Z \to Y$, there exist a morphism of $\mathcal{K}$, $v : W \to Z$ and a morphism in S, $t : W \to X$, such that the following diagram is commutative.

$$\begin{tikzcd} W \arrow[r, "v"] \arrow[d, "t"] & Z \arrow[d, "s"] \\ X \arrow[r, "u"] & Y \end{tikzcd}$$

- Given two morphisms of $\mathcal{K}$, $u, v : X \to Y$, there exist a morphism in S, $s : Y \to Y'$, such that $s \circ u = s \circ v$ if and only if there exist a morphism in S, $t : X' \to X$, such that $u \circ t = v \circ t$.

Definition 2.1.6. Given a category $\mathcal{K}$ and a multiplicative system S, the *localization* of $\mathcal{K}$ with respect to S is the category $\mathcal{K}_S$ such that

- the objects of $\mathcal{K}_S$ are the objects of $\mathcal{K}$,
- the morphisms $\mathrm{Hom}(X, Y)$ of $\mathcal{K}_S$ are the equivalence classes of the diagrams

$$\begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr, "u"] & \\ X & & Y \end{tikzcd}$$

where $s \in S$ and u is a morphism of $\mathcal{K}$, by the following equivalence relation.

$$\begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr, "u"] & \\ X & & Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & X'' \arrow[dl, "s'"'] \arrow[dr, "u'"] & \\ X & & Y \end{tikzcd}$$

are equivalent if there exist an object of $\mathcal{K}$, Z, an element of S, $t : Z \to X$, and morphisms of $\mathcal{K}$, $v : Z \to X'$ and $w : Z \to X''$ such that the following diagram is commutative:

$$\begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr, "u"] & \\ X & Z \arrow[l, "t"] \arrow[u, "v"] \arrow[d, "w"] & Y \\ & X'' \arrow[ul, "s'"] \arrow[ur, "u'"'] & \end{tikzcd}$$

Now, we define the composition of morphisms in $\mathcal{K}_S$. Consider two diagrams

$$\begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr, "u"] & \\ X & & Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & Y' \arrow[dl, "t"'] \arrow[dr, "v"] & \\ Y & & Z \end{tikzcd} \tag{2.1}$$

By the third property of Definition 2.1.5 there exist a morphism of $\mathcal{K}$, $w : X'' \to Y'$, and a morphism in S, $s' : X'' \to X$, such that the diagram

$$\begin{tikzcd} & & X'' \arrow[dl, "s'"'] \arrow[dr, "w"] & & \\ & X' \arrow[dl, "s"'] \arrow[dr, "u"] & & Y' \arrow[dl, "s"] \arrow[dr, "v"] & \\ X & & Y & & Z \end{tikzcd}$$

is commutative. Then, the composition of the equivalence classes of the diagrams (2.1) is the equivalence class of

$$\begin{tikzcd} & X'' \arrow[dl, "s \circ s'"'] \arrow[dr, "v \circ w"] & \\ X & & Z \end{tikzcd}$$

The localization of a category verifies a universal property as we explain now.
Let us consider the canonical functor $Q : \mathcal{K} \to \mathcal{K}_S$ such that

- if X is an object of $\mathcal{K}$, then $Q(X) := X$,
- if $u : X \to Y$ is a morphism in $\mathcal{K}$, then $Q(u)$ is the equivalence class of

$$\begin{tikzcd} & X \arrow[dl, "Id_X"'] \arrow[dr, "u"] & \\ X & & Y \end{tikzcd}$$

Then, if $s : X \to Y$ is in S, $Q(s)$ is an isomorphism. It is easy to check that the inverse morphism is the equivalence class of

$$\begin{tikzcd} & X \arrow[dl, "s"'] \arrow[dr, "Id_X"] & \\ Y & & X \end{tikzcd}$$

Moreover, if $\mathcal{F} : \mathcal{K} \to \mathcal{L}$ is another functor of categories such that $\mathcal{F}(s)$ is an isomorphism for every $s \in S$, $\mathcal{F}$ factors uniquely through Q ([Ban07, Proposition 2.4.3]).

If the category $\mathcal{K}$ is triangulated and the multiplicative system verifies some properties, we can also give a triangulated structure to the localization. The following definition explain what are the conditions which the multiplicative system should verify.

Definition 2.1.7. A multiplicative system in $\mathcal{K}$, S, is *compatible with the triangulation* of $\mathcal{K}$ if it verifies the following properties.

- A morphism of $\mathcal{K}$, s, is in S if and only if $T(s)$ is in S, where T is the automorphism fixed in $\mathcal{K}$.
- Given two distinguished triangles (X, Y, Z, u, v, w) and (X', Y', Z', u', v', w') and a commutative diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "s"] & Y \arrow[d, "t"] \\ X' \arrow[r, "u'"] & Y' \end{tikzcd}$$

  such that s and t are in S, there exist a morphism in S, $\gamma : Z \to Z'$ which complete the morphism between the triangles.

If the category $\mathcal{K}$ is triangulated and S is a multiplicative system compatible with the triangulation, $\mathcal{K}_S$ is also triangulated with the triangulation induced by Q (a triangle in $\mathcal{K}_S$ is distinguished it and only if it is isomorphic to the image by Q of some distinguished triangle).

Let us suppose $\mathcal{K}$ is the homotopic category $\mathcal{K}(\mathcal{A})$. The set of quasi-isomorphisms of $\mathcal{K}(\mathcal{A})$, Qis, is a multiplicative system compatible with the triangulation (see [Ban07, Proposition 2.4.7]). We can introduce now the principal definition of this section:

Definition 2.1.8. The *derived category* of $\mathcal{A}$, $D(\mathcal{A})$, is the localization of the category $\mathcal{K}(\mathcal{A})$ with respect to Qis.

So, $D(\mathcal{A})$ is a triangulated category with the triangulation induced by $Q : \mathcal{K}(\mathcal{A}) \to D(\mathcal{A})$. From now on, every triangle in the document is assumed to be distinguished unless otherwise stated.

The following remark is essential for many proofs along the thesis.

Remark 2.1.9. The triangles in $D(\mathcal{A})$ induce long exact sequences of hypercohomology and cohomology, that is, if

$$A^\bullet \to B^\bullet \to C^\bullet \xrightarrow{[1]}$$

is a distinguished triangle in $D(\mathcal{A})$, $\mathcal{H}^i$ denotes the cohomology sheaf and $\mathbb{H}^i$ denotes the hypercohomology presheaf, there are long exact sequences

$$... \to \mathcal{H}^{i-1}(C^\bullet) \to \mathcal{H}^i(A^\bullet) \to \mathcal{H}^i(B^\bullet) \to \mathcal{H}^i(B^\bullet) \to \mathcal{H}^{i+1}(A^\bullet) \to ...$$

and

$$... \to \mathbb{H}^{i-1}(C^\bullet) \to \mathbb{H}^i(A^\bullet) \to \mathbb{H}^i(B^\bullet) \to \mathbb{H}^i(B^\bullet) \to \mathbb{H}^{i+1}(A^\bullet) \to ...$$

The reader can check [Ban07, Section 2.3] for more details and proofs about this remark.

Now, we give some notions about derived functors. Let us denote by $\mathcal{K}^+(\mathcal{A})$ the full subcategory of $\mathcal{K}(\mathcal{A})$ given by bounded below complexes. Then, the bounded below derived category of $\mathcal{A}$, $D^+(\mathcal{A})$, is the localization of $\mathcal{K}^+(\mathcal{A})$ with respect to Qis.

By [Ban07, Lemma 2.4.18], if $\mathcal{I}$ is the full subcategory of injective objects of $\mathcal{A}$, all the quasi-isomorphisms in $\mathcal{K}^+(\mathcal{I})$ are isomorphisms. So, the derived category $D^+(\mathcal{I})$ is equal to the homotopic category $\mathcal{K}^+(\mathcal{I})$. Moreover, if $\mathcal{A}$ has enough injectives, every bounded below complex in $\mathcal{A}$ is quasi-isomorphic to a bounded below complex of injective objects. Hence, there is a equivalence of categories between $D^+(\mathcal{A})$ and $D^+(\mathcal{I}) = \mathcal{K}^+(\mathcal{I})$ (see [Ban07, Theorem 2.4.19]).

Given a functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ between abelian categories, we have an induced functor $\bar{\mathcal{F}}$ between the homotopic categories

$$\bar{\mathcal{F}} : \mathcal{K}(\mathcal{A}) \to \mathcal{K}(\mathcal{B})$$

This functor induces a morphism between the derived categories $D(\mathcal{A}) \to D(\mathcal{B})$ if and only if it transform quasi-isomorphisms into quasi-isomorphisms. However, if $\mathcal{A}$ has enough injectives we can always define the *right derived functor* of $\mathcal{F}$. as the following composition

$$R\mathcal{F} : D^+(\mathcal{A}) \cong \mathcal{K}^+(\mathcal{I}) \xrightarrow{\bar{\mathcal{F}}} K(\mathcal{B}) \xrightarrow{Q} D(B).$$

The following two definitions about derived functors are also important and they will be use along the thesis.

Definition 2.1.10. If H^i denotes the i-th cohomology functor, the *i-th right derived functor* of $\mathcal{F}$ is $R^i\mathcal{F} = H^i \circ R\mathcal{F}$.

Definition 2.1.11. An object $A^\bullet$ of $D^+(\mathcal{A})$ is *$\mathcal{F}$-acyclic* if $R^i\mathcal{F}(A^\bullet)$ is equal to 0 for every i.

Remark 2.1.12. Let $\mathcal{K}^b(\mathcal{A})$ be the full subcategory of $\mathcal{K}^+(\mathcal{A})$ given by bounded complexes and let $D^b(\mathcal{A})$ be the localization of $\mathcal{K}^b(\mathcal{A})$ with respect to the set of quasi-isomorphisms. Since $D^b(\mathcal{A})$ is a full subcategory of $D^+(\mathcal{A})$ the right derived functors are defined in $D^b(\mathcal{A})$.

For simplicity, we will denote the derived right functor $R\mathcal{F}$ by $\mathcal{F}$.

2.2 Preliminaries on Spectral Sequences

In this section, we fix some notation about spectral sequences and we describe the local to global spectral sequence, a particular case of the Grothendieck spectral sequence which we use along the thesis.

First we give some basic definitions in order to fix notation.

Definition 2.2.1. A *spectral sequence* in an abelian category $\mathcal{A}$ is composed by

- a family of objects of $\mathcal{A}$, $\{E_r^{p,q}\}$, defined for all integers p and q and every integer r greater or equal than a fixed a in $\mathbb{Z}_{\geq 0}$
- morphisms $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ such that the composition $d_r^{p+r,q-r+1} \circ d_r^{p,q}$ is equal to 0

which verify that

$$E_{r+1}^{p,q} \cong Ker(d_r^{p,q})/Im(d_r^{p-r,q+r-1}).$$

Definition 2.2.2. The *total degree* of the element $E_r^{p,q}$ of a spectral sequence is $p+q$.

Definition 2.2.3. A spectral sequence is *bounded* if for every integer n, there are finitely many elements with degree n in the spectral sequence.

Given a bounded spectral sequence, there exists an integer r_0 such that the morphisms $d_r^{p,q}$ are 0 for every $r \geq r_0$ and every pair of integers p, q. So, we have an equality $E_r^{p,q} = E_{r_0}^{p,q}$ for every $r \geq r_0$ and every $p, q \in \mathbb{Z}$. Then, we denote the objects $E_{r_0}^{pq}$ by E_∞^{pq} and we say the spectral sequence *degenerates* at r_0.

These definitions allow us give a notion of convergence for bounded spectral sequences.

Definition 2.2.4. We say that a bounded spectral sequence $E_r^{p,q}$ *converges* to $H^* = \{H^n\}_{n\in\mathbb{Z}}$ if there exist filtrations

$$0 = F^t H^n \subseteq F^{t-1} H^n \subseteq ... \subseteq F^s H^n = H^n$$

such that $E_\infty^{p,q} \cong F^p H^{p+q} / F^{p+1} H^{p+q}$.

Then, we write $E_r^{p,q} \Rightarrow H^{p+q}$.

Now, we can recall the Grothendieck spectral sequence.

Theorem 2.2.5. *Let $\mathcal{G} : \mathcal{A} \to \mathcal{B}$ and $\mathcal{F} : \mathcal{B} \to \mathcal{C}$ be two left exact additive functors between abelian categories. Moreover, suppose $\mathcal{A}$ and $\mathcal{B}$ have enough injectives and $\mathcal{G}$ transform injective objects of $\mathcal{A}$ into $\mathcal{F}$-acyclic objects. Then, for every object A of $D^+(\mathcal{A})$, there exist a spectral sequence*

$$E_r^{p,q} \Rightarrow R^{p+q}(F \circ G)(A)$$

such that $E_2^{p,q} = (R^p F \circ R^q G)(A)$.

This spectral sequence is the Grothendieck spectral sequence for functors $\mathcal{G}$ and $\mathcal{F}$.

For a proof of this theorem and an explicit desctiption of the Grothendieck spectral sequence, see [**?**, Theorem 5.8.3].

Let us consider a particular case. Suppose $\mathcal{A} = \mathcal{B}$ is the category of sheaves of $\mathbb{Q}$-vector spaces on a topological space X, $\mathcal{G}$ is the identity and $\mathcal{F}$ is the functor of global sections, $\Gamma(X, \;)$. In this case, for any bounded below complex of sheaves $A^\bullet$, $R^{p+q}(F \circ G)(A^\bullet)$ is equal to the $(p+q)$-th hypercohomology group $\mathbb{H}^{p+q}(X, A^\bullet)$ and $E_2^{p,q} = \mathbb{H}^p(X, \mathcal{H}^q(A^\bullet))$ where $\mathcal{H}^q(A^\bullet)$ denotes the q-th cohomology sheaf of $A^\bullet$. We will call this spectral sequence the *local to global spectral sequence* of $A^\bullet$.

2.3 Preliminaries on intersection cohomology

In this section, we recall the notions concerning the intersection cohomology theory that we will use.

We describe the intersection cohomology theory on a class of topological spaces endowed with a stratification, the topological pseudomanifolds. We use a definition of topological pseudomanifold similar to [Ban07, Definition 4.1.1].

Definition 2.3.1. A *topological pseudomanifold* is a paracompact Hausdorff topological space with a filtration by closed subspaces

$$X = X_d \supset X_{d-2} \supset ... \supset X_0 \supset X_{-1} = \emptyset.$$

such that

- Each pair (X_i, X_{i-1}) is a locally finite relative CW-complex.

- Every non-empty $X_{d-k} \setminus X_{d-k-1}$ is a topological manifold of dimension $d-k$ called pure stratum of X.
- $X \setminus X_{d-2}$ is dense in X.
- **Local normal triviality.** For each point $x \in X_{d-k} \setminus X_{d-k-1}$, there exists an open neighborhood U of x in X, a compact topological pseudomanifold L of dimension $k-1$ with stratification

  $$L = L_{k-1} \supset L_{k-3} \supset ... \supset L_0 \supset L_{-1} = \emptyset$$

  and a homeomorphism

  $$\varphi : U \xrightarrow{\cong} \mathbb{R}^{d-k} \times c^{\circ}(L)$$

  where $c^{\circ}(L)$ is the open cone of L, such that it preserve the strata, that is, $\varphi(U \cup X_{d-r}) = \mathbb{R}^{d-k} \times c^{\circ}(L_{k-r-1})$.

  L is called the link of the point x.

The idea of the intersection cohomology theory is computing cohomology considering only (singular, simplicial, CW, etc.) chains which intersect "adequately" with the strata. With this purpose, we introduce a parameter called perversity that specifies the "allowable deviation" from transversality.

Definition 2.3.2. A perversity is a map $\bar{p} : \mathbb{Z}_{\geq 2} \to \mathbb{Z}_{\geq 0}$ such that $\bar{p}(2) = 0$ and $\bar{p}(k) \leq \bar{p}(k+1) \leq \bar{p}(k) + 1$.

Some special perversities are

- The zero perversity, $\bar{0}(k) = 0$.
- The total perversity, $\bar{t}(k) = k - 2$.
- The lower middle perversity, $\bar{m}(k) = \lfloor \frac{k}{2} \rfloor - 1$.
- The upper middle perversity, $\bar{n}(k) = \lceil \frac{k}{2} \rceil - 1$.
- Given a perversity $\bar{p}$, the complementary perversity is $\bar{t} - \bar{p}$. It is usually denoted by $\bar{q}$.

The lower and the upper middle perversities are complementary.
Let X be a topological pseudomanifold with stratification

$$X = X_d \supset X_{d-2} \supset ... \supset X_0 \supset X_{-1} = \emptyset.$$

Let $U_k := X \setminus X_{d-k}$ and let $i_k : U_k \to U_{k+1}$ and $j_k : X_{d-k} \setminus X_{d-k-1} \to U_{k+1}$ be the usual inclusions.

Let us denote by $D^b_{cc}(X)$ the bounded derived category of cohomologically constructible sheaves of rational vector spaces on X with the previous stratification. Moreover, we denote by $\mathbb{Q}_X$ the constant sheaf on X with rational coefficients.

The following notions are needed for our construction.

Definition 2.3.3. A *local system* on X is a locally constant sheaf of $\mathbb{Q}$-vector spaces (for the Euclidean topology) having finite dimensional stalks. A local system is *simple* if has no non-trivial local subsystems (i.e., simple in the category of local systems) and *semisimple* if it is direct sum of simple local systems.

Let $\mathscr{L}$ be a local system defined on an open dense subset of $U \subset X \setminus X_{d-2}$. By changing the stratification, if it is necessary, we can suppose $\mathscr{L}$ is defined on U_2 (we can give a new stratification such that $X'_{d-2} := X \setminus U$ and $X'_{d-2} \supset X'_{d-3} \supset ... \supset X'_0 \supset \emptyset$ is a stratification of X'_{d-2}).

We use the definition of intersection cohomology complex given in [GM83, Section 3.1].

Definition 2.3.4. The *intersection cohomology complex* of X with coefficient $\mathscr{L}$ and perversity $\bar{p}$, $IC_{\bar{p}X}(\mathscr{L})$, is

$$\tau_{\leq \bar{p}(d)-d} i_{d*} \tau_{\leq \bar{p}(d-1)-d} i_{d-1*} \cdots \tau_{\leq \bar{p}(2)-d} i_{2*} \mathscr{L}[d]$$

where $\tau_{\leq}$ denotes the usual functor of truncation.

Moreover, the intersection cohomology complex of X with coefficient $\mathscr{L}$ and perversity $\bar{p}$ is the unique complex of sheaves (up to isomorphism of the derived category) which verifies the axioms $[AX1]$ of [GM83, section 3.3]:

(a) $IC_{\bar{p}X}(\mathscr{L})_{|U_2}$ is isomorphic to $\mathscr{L}[d]$ in the derived category.

(b) The cohomology sheaf $\mathcal{H}^i(IC_{\bar{p}X}(\mathscr{L}))$ is 0 if $i < -d$,

(c) For $k = 2, ..., d$ $\mathcal{H}^i(j_k^* IC_{\bar{p}X}(\mathscr{L})_{|U_{k+1}})$ is equal to 0 if $i > \bar{p}(k) - d$,

(d) For $k = 2, ..., d$ the natural morphism

$$\mathcal{H}^i(j_k^* IC_{\bar{p}X}(\mathscr{L})_{|U_{k+1}}) \to \mathcal{H}^i(j_k^* i_{k*} IC_{\bar{p}X}(\mathscr{L})_{|U_k})$$

is an isomorphism if $i \leq \bar{p}(k) - d$.

If $\mathscr{L}$ is the constant sheaf $\mathbb{Q}_{U_2}$, then we denote $IC_{\bar{p}X} := IC_{\bar{p}X}(\mathscr{L})$.

Although the stratification of X plays an important role in our definition of intersection cohomology complex, we have the following result.

Theorem 2.3.5. *The intersection cohomology complex of a topological pseudomanifold X is independent of the stratification.*

Section 4 of [GM83] is devoted to prove this theorem. Moreover, the following equivalent definition of intersection cohomology complex which appear in [CaMi, Section 3.8] does not use any stratification.

Definition 2.3.6. Let $\mathscr{L}$ be a local system defined on an open dense subset U of the regular locus of X and let $\gamma : U \to X$ be the inclusion.

The *intersection cohomology complex* with coefficient $\mathscr{L}$ and perversity $\bar{p}$, $IC_{\bar{p}\,X}(\mathscr{L})$, is the image of the natural morphism

$$^{\bar{p}}H^0(\gamma_!\mathscr{L}[\dim(X)]) \to {}^{\bar{p}}H^0(\gamma_*\mathscr{L}[\dim(X)])$$

Here $^{\bar{p}}H^0$ is the (zero-th) $\mathfrak{t}$-cohomology functor for the natural $\mathfrak{t}$-structure on $D^b_{cc}(Y;\mathbb{Q})$ defining the category of perverse sheaves.

We refer the reader to [BBD82] anc [CaMi] for the theory of perverse sheaves.

Now, we can define the intersection cohomology.

Definition 2.3.7. The i-th *rational intersection cohomology group* $IH^i_{\bar{p}}(X;\mathscr{L})$ with coefficient $\mathscr{L}$ and perversity $\bar{p}$ is the $(i-d)$-th hypercohomology group of the intersection cohomology complex $\mathbb{H}^{i-d}(IC_{\bar{p}\,X}(\mathscr{L}))$.

We denote $IH^i_{\bar{p}}(X;\mathbb{Q}) := IH^i_{\bar{p}}(X;\mathscr{L})$ if $\mathscr{L}$ is the constant sheaf.

The intersection cohomology complex verify the following condition of duality (see [GM83, Section 5]).

Theorem 2.3.8. *Let $\bar{p}$ and $\bar{q}$ be two complementary perversities. If $IC_{\bar{p}\,X}(\mathscr{L})$ is the intersection cohomology complex of X with coefficient $\mathscr{L}$ and perversity $\bar{p}$, then $\mathcal{D}IC_{\bar{p}\,X}(\mathscr{L})[d]$, where $\mathcal{D}$ denotes the Verdier dual, is the intersection cohomology complex of X with coefficient $\mathcal{D}\mathscr{L}[d]$ and perversity $\bar{q}$.*

In particular, when $\mathscr{L}$ is the constant sheaf we have $\mathcal{D}IC_{\bar{p}\,X}[d] = IC_{\bar{q}\,X}$.

Consequently, if $\mathscr{L}_1$ and $\mathscr{L}_2$ are two local systems such that $\mathscr{L}_2 = \mathcal{D}\mathscr{L}_1[d]$, we obtain the following version of Poincaré duality:

$$IH^i_{\bar{p}}(X;\mathscr{L}_1) \cong IH^{d-i}_{\bar{q}}(X;\mathscr{L}_2)^v$$

In particular, $IH^i_{\bar{p}}(X;\mathbb{Q}) \cong IH^{d-i}_{\bar{q}}(X;\mathbb{Q})^v$.

Definition 2.3.9. The *i-th intersection cohomology Betti number* is $b_i(X) := \dim IH^i(X;\mathbb{Q})$.

We have the equality

$$b_i(X) = 0 \quad \text{if} \quad i < 0 \quad \text{or} \quad i > d.$$

Definition 2.3.10. The *Poincaré polynomial* of X is

$$P_X(t) := \sum_{i=0}^{2d} b_i(X)\, t^i$$

In the particular case where X is rationally smooth, rational intersection cohomology coincides with classical rational cohomology. Now, we explain how compute the rational cohomology groups of every smooth projective variety.

Definition 2.3.11. The *Hodge–Deligne polynomial* of a variety X is

$$E(X;u,v)=\sum_{p,q=0}^{d}\sum_{i=0}^{2d}(-1)^i h^{p,q}(H^i_c(X;\mathbb{C}))u^p v^q\in\mathbb{Z}[u,v],$$

where $d=\dim(X)$ and $h^{p,q}(H^i_c(X;\mathbb{C}))$ is the dimension of the (p,q)-type Hodge component in the i-th cohomology group $H^j_c(X;\mathbb{C})$ with compact support.

The polynomial $E(\quad;u,v)$ satisfies the following properties:

(i) If Z is a Zariski closed subset of X and $U=X\setminus Z$, then

$$E(X;u,v)=E(Z;u,v)+E(U;u,v).$$

(ii) If X_1 and X_2 are two varieties, then

$$E(X_1\times X_2;u,v)=E(X_1;u,v)\cdot E(X_2;u,v).$$

(iii) If X is smooth and projective, then $P_X(t)=E(X;-t,-t)$.

For instance, if Y is a smooth projective curve of genus g, then

$$E(Y;u,v)=uv-g(u+v)+1.$$

Hence $E(\mathbb{P}^1;u,v)=uv+1$, $E(\mathbb{A}^1;u,v)=uv$, $E(\mathbb{C}^*;u,v)=uv-1$ and so

$$E((\mathbb{C}^*)^r;u,v)=(uv-1)^r \text{ for every } r\in\mathbb{Z}_{\geq 0}.$$

Remark 2.3.12. One can also define the intersection cohomology E-polynomial of a singular quasi-projective variety in a similar way as the Hodge-Deligne polynomial for classical cohomology. We refer the reader to [BB96, Section 3] for some properties.

We need the following technical lemma. It is a consequence of results in [BBD82, Section 4.2].

Lemma 2.3.13. *Let $f:X\to Y$ be an étale morphism between two varieties X and Y. Then $f^*IC_{\bar{p}Y}$ identifies with the intersection cohomology complex $IC_{\bar{p}X}$.*

Proof. Let $d=\dim(X)$. If U is a smooth open dense subset of Y and $\gamma:U\to Y$ is the inclusion, then recall that $IC_{\bar{p}Y}$ is the image of the natural morphism

$${}^{\bar{p}}H^0(\gamma_!\mathbb{Q}_U[d])\to {}^{\bar{p}}H^0(\gamma_*\mathbb{Q}_U[d]).$$

Since the functor f^* is exact (see [BBD82, p.108-109]), the complex $f^*IC_{\bar{p}Y}$ is the image of the induced morphism

$$^{\bar{p}}H^0(f^*\gamma_!\mathbb{Q}_U[d]) \to {}^{\bar{p}}H^0(f^*\gamma_*\mathbb{Q}_U[d]).$$

Let us consider the Cartesian diagram

$$\begin{tikzcd} f^{-1}(U) \arrow[r, "f"] \arrow[d, "\beta"] & U \arrow[d, "\gamma"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where $\beta : f^{-1}(U) \to X$ is the inclusion. Note that $f^{-1}(U)$ is again a smooth dense open subset of X. By base change we have the isomorphisms

$$f^*\gamma_!\mathbb{Q}_U[d] \simeq \beta_! f^*\mathbb{Q}_U[d] \simeq \beta_!\mathbb{Q}_{f^{-1}(U)}[d],$$

in the derived category $D^b_{cc}(X;\mathbb{Q})$. Since f is étale, we have $f^* = f^!$ (see [BBD82, Section 4.2.4]), hence by base change $f^*\gamma_*\mathbb{Q}_U[d] \simeq \beta_*\mathbb{Q}_{f^{-1}(U)}[d]$. Thus we obtain that $f^*IC_{\bar{p}Y}$ is the image of

$$^{p}H^0(\beta_!\mathbb{Q}_{f^{-1}(U)}[d]) \to {}^{p}H^0(\beta_*\mathbb{Q}_{f^{-1}(U)}[d])$$

which coincides with $IC_{\bar{p}X}$. $\square$

2.3.1 Intersection cohomology theory on complex varieties

We do not need a general version of the following results. So, to simplify the notation and ease consulting bibliography, we make the following assumptions for the rest of the section:

- All the spaces are complex varieties. Then, the upper middle perversity and the lower middle perversity (see Definition 2.3.2) are equal over the codimension of the strata. We suppose the perversity is the middle perversity and we erase it in the notation.

- Since the spaces are complex varieties, we can consider the complex dimension of X instead of the real one in the definition of intersection cohomology complex. This is the usual convention when the intersection cohomology theory is developed over complex varieties and it simplifies the notation. So, it is the convention we use along the rest of the section and along Chapter 3. Then,

  – the intersection cohomology complex with coefficient $\mathscr{L}$, $IC_X(\mathscr{L})$, is the image of the natural morphism

  $$^{\bar{p}}H^0(\gamma_!\mathscr{L}[\dim_{\mathbb{C}}(X)]) \to {}^{\bar{p}}H^0(\gamma_*\mathscr{L}[\dim_{\mathbb{C}}(X)]).$$

- the i-th rational intersection cohomology group $IH^i(X;\mathscr{L})$ with coefficient $\mathscr{L}$ is the hypercohomology group $\mathbb{H}^{i-d}(IC_{\bar{p}\,X}(\mathscr{L}))$ where $d = \dim_{\mathbb{C}}(X)$.
- by properties of the Verdier dual, the shift which appear in Theorem 2.3.8 is not needed anymore, that is, $\mathcal{D}IC_X(\mathscr{L})$ is equal to $IC_X(\mathcal{D}\mathscr{L})$. In particular, the complex IC_X (where $\mathscr{L}$ is the constant sheaf) is autodual.

We enunciate the decomposition theorem of Beilinson–Bernstein–Deligne–Gabber which allows us to describe the topology of singular proper algebraic maps (see [BBD82, Theorem 6.25]).

Theorem 2.3.14. *Let $f : X \to Z$ be a proper (algebraic) morphism between varieties X and Z. Then there exists a finite family $(Z_\alpha, \mathscr{L}_\alpha, d_\alpha)$ where for every index α, $Z_\alpha \subseteq Z$ is a smooth irreducible (algebraic) subvariety which is locally closed (for the Zariski topology), $\mathscr{L}_\alpha$ is a semisimple local system on Z_α and $d_\alpha \in \mathbb{Z}$ such that we have an isomorphism*

$$f_*\, IC_X \simeq \bigoplus_\alpha (i_\alpha)_*\, IC_{\bar{Z}_\alpha}(\mathscr{L}_\alpha)[-d_\alpha]$$

in the derived category $D^b(Z;\mathbb{Q})$, where $\bar{Z}_\alpha$ is the Zariski closure of Z_α in Z and $i_\alpha : \bar{Z}_\alpha \to Z$ is the inclusion. This implies that for every Euclidean open subset $U \subseteq Z$ and for every $j \in \mathbb{Z}$, we have an isomorphism of $\mathbb{Q}$-vector spaces

$$IH^j(f^{-1}(U);\mathbb{Q}) \simeq \bigoplus_\alpha IH^{j-l_\alpha}(U \cap \bar{Z}_\alpha; \mathscr{L}_\alpha),$$

where $l_\alpha = \dim(X) - \dim(Z_\alpha) + d_\alpha$.

In [CMM15] the preceding theorem was made explicit in the case where X and Z are toric varieties and f is a proper toric map. For convenience we recall this result.

Theorem 2.3.15. *Let X, Z be toric varieties and let Σ_Z be the fan defining Z. For every $\tau \in \Sigma_Z$, denote by $V(\tau)$ the corresponding orbit closure. Consider a birational proper toric map $f : X \to Z$ given by a subdivision of fans. Then we have an isomorphism*

$$f_*\, IC_X \simeq \bigoplus_{\tau \in \Sigma_Z} \bigoplus_{b \in \mathbb{Z}} (i_\tau)_*\, IC_{V(\tau)}^{\oplus s_{\tau,b}}[-b]$$

in the derived category $D^b(Z;\mathbb{Q})$, where $i_\tau : V(\tau) \to Z$ is the inclusion. This implies that for every Euclidean open subset $U \subseteq Z$ and every $j \in \mathbb{Z}$, we have an isomorphism of $\mathbb{Q}$-vector spaces

$$IH^j(f^{-1}(U);\mathbb{Q}) \simeq \bigoplus_{\tau \in \Sigma_Z} \bigoplus_{b \in \mathbb{Z}} IH^{j-l_{\tau,b}}(U \cap V(\tau);\mathbb{Q})^{\oplus s_{\tau,b}},$$

where $l_{\tau,b} = b + \dim(X) - \dim(V(\tau))$. The sequence $s_{\tau,b}$ of natural numbers depends on the fan subdivision and satisfies the following conditions.

(i) $s_{\tau,b} = s_{\tau,-b}$ *for all* $\tau \in \Sigma_Z$ *and* $b \in \mathbb{Z}$.

(ii) *If* f *is a projective morphism, then* $s_{\tau,b} \geq s_{\tau,b+2l}$ *for all* $\tau \in \Sigma_Z$ *and all* $b, l \in \mathbb{Z}_{\geq 0}$.

(iii) $s_{\tau,b} = 0$ *if* $l_{\tau,b}$ *is odd.*

(iv) $s_{0,0} = 1$ *and* $s_{0,b} = 0$ *for every* $b \in \mathbb{Z} \setminus \{0\}$.

Note that assertion (iv) in Theorem 2.3.15 is a consequence of [CMM15, Remark 7.1, Theorems 7.2,7.4]. The following terminology will be useful for the sequel.

Definition 2.3.16. With the same notation as in 2.3.15, let us denote by Σ_X the fan defining the toric variety X. The sequence of coefficients $s_{\tau,b}$ (τ and b run respectively Σ_Z and $\mathbb{Z}$) involving in Theorem 2.3.15 will be called an *s-sequence* of the subdivision Σ_X of Σ_Z.

Let us end this section by recalling the notion of h-vectors which describes the intersection cohomology Betti numbers of projective toric varieties (see [Sta87, Theorem 3.1] and [DL91, Fie91]).

We will consider here the empty set $\emptyset$ as a common face of every polytope. Let us define two polynomials $h(\,*\,, t), g(\,*\,, t) \in \mathbb{Z}[t]$ by induction [Sta87, Section 2], where the first entry $*$ depends on a polytope. Let $g(\emptyset, t) = 1$ and $\dim(\emptyset) = -1$. Assume that $g(\,*\,, t)$ is defined for every polytope of dimension $< d$. Then for a polytope Q of dimension d, we let

$$h(Q, t) = \sum_{\Lambda \subsetneq Q} g(\Lambda, t)(t-1)^{d-1-\dim(\Lambda)},$$

where the sum runs through the set of proper faces of Q (included the empty set $\emptyset$). Let us denote by $h_k(Q)$ the k-th coefficient of $h(Q, t)$. One defines $g(Q, t)$ by the relation

$$g(Q, t) = \sum_{k=0}^{\lfloor d/2 \rfloor} (h_k(Q) - h_{k-1}(Q))t^k \text{ with } h_{-1}(Q) = 0.$$

The polynomial $h(Q, t)$ is called the *h-polynomial* of Q and $(h_0(Q), \dots, h_d(Q))$ is the *h-vector* of Q. More generally, we will let $h_j(Q) = 0$ for any $j \in \mathbb{Z}_{<0}$. We refer to [Fie91] for a geometric meaning of $h(\,*\,, t)$ and $g(\,*\,, t)$.

Theorem 2.3.17. *Let* X *be a projective toric variety defined by a fan* Σ_X. *Let us fix a polytope* $Q \subseteq M_{\mathbb{Q}}$ *such that* Σ_X *is the normal fan of* Q. *Then we have the equality* $P_X(t) = h(Q, t^2)$.

2.4 Preliminaries on $\mathbb{T}$-varieties of complexity one

In this section, we recall some notions on algebraic torus actions of complexity one (see [AH06, AHS08, Tim08, Lan14] for details). First we give some basic definitions.

Definition 2.4.1. Let $\mathbb{T} \cong (\mathbb{C}^*)^n$ be an algebraic torus. A $\mathbb{T}$-*variety* is a normal variety endowed with an effective $\mathbb{T}$-action.

Definition 2.4.2. The *complexity* of a $\mathbb{T}$-variety X is the non-negative number $\dim(X) - \dim(\mathbb{T})$; it corresponds to the transcendence degree over $\mathbb{C}$ of the field extension $\mathbb{C}(X)^{\mathbb{T}}$ of invariant rational functions on X.

By a result of Rosenlicht (see [Ros63]), the complexity of X is the codimension of a general $\mathbb{T}$-orbit.

In particular, a toric variety is a $\mathbb{T}$-variety of complexity 0. We are interested in $\mathbb{T}$-varieties of complexity one. In this case, the inclusion

$$\mathbb{C}(X)^{\mathbb{T}} \subseteq \mathbb{C}(X)$$

yields a map

$$\pi : X \dashrightarrow Y$$

to a smooth projective curve Y. Here Y is the projective curve obtained from the algebraic function field $\mathbb{C}(X)^{\mathbb{T}}$ of one variable. The map π is called the *rational quotient* of X.

Definition 2.4.3. We say that the complexity-one $\mathbb{T}$-variety X is *contraction-free* if the rational quotient map given by the $\mathbb{T}$-action on X is a regular morphism.

Now, we recall the combinatorial description of a $\mathbb{T}$-variety with complexity one. Let us consider the lattice of characters M and lattice of one-parameter subgroups N. Then the duality between M and N extends naturally to a duality

$$M_{\mathbb{Q}} \times N_{\mathbb{Q}} \to \mathbb{Q},\ (m, v) \mapsto \langle m, v \rangle$$

between the $\mathbb{Q}$-vector spaces $M_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} M$ and $N_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} N$.

According to the Sumihiro theorem (see [Sum74, Section 3, Corollary 2]), every $\mathbb{T}$-variety is covered by affine $\mathbb{T}$-stable Zariski open subsets. Thus we recall first how to describe an affine $\mathbb{T}$-variety by combinatorial objects in the setting of the complexity one.

Definition 2.4.4. A polyhedral cone $\sigma \subseteq N_{\mathbb{Q}}$ is said to be *strongly convex* if it contains no lines; this condition is equivalent to the *dual cone*

$$\sigma^{\vee} = \{m \in M_{\mathbb{Q}} \,|\, \forall v \in \sigma,\ \langle m, v \rangle \geq 0\}$$

being full dimensional.

Let us fix a strongly convex polyhedral cone $\sigma \subseteq N_{\mathbb{Q}}$. A σ-*polyhedron* of $N_{\mathbb{Q}}$ is a Minkowski sum $Q + \sigma$, where $Q \subseteq N_{\mathbb{Q}}$ is a polytope (i.e., Q is the convex hull of a non-empty finite subset of $N_{\mathbb{Q}}$). Let Y be a smooth curve.

Definition 2.4.5. A *σ-polyhedral divisor* $\mathfrak{D}$ on the curve Y is a formal sum

$$\mathfrak{D} = \sum_{y \in Y} \mathfrak{D}_y \cdot [y],$$

where every $\mathfrak{D}_y$ is a σ-polyhedron and $\mathfrak{D}_y = \sigma$ for all but finitely many $y \in Y$.

For every $m \in \sigma^\vee$ we define a $\mathbb{Q}$-divisor on Y by letting

$$\mathfrak{D}(m) = \sum_{y \in Y} \min_{v \in \mathfrak{D}_y} \langle m, v \rangle \cdot [y].$$

The curve Y (respectively the cone σ) is usually called the *locus* (respectively the *tail*) of $\mathfrak{D}$.

Definition 2.4.6. The σ-polyhedral divisor $\mathfrak{D}$ is called *proper* if Y is affine; or Y is projective and $\mathfrak{D}$ verifies the additional properties:

(i) The *degree* $\deg(\mathfrak{D}) := \sum_{y \in Y} \mathfrak{D}_y$ is strictly contained in σ.

(ii) For every $m \in \sigma^\vee$ such that $\min_{v \in \deg(\mathfrak{D})} \langle m, v \rangle = 0$, the divisor $\mathfrak{D}(dm)$ is principal for some $d \in \mathbb{Z}_{>0}$.

We denote by $\operatorname{PPDiv}_{\mathbb{Q}}(Y, \sigma)$ the set of proper σ-polyhedral divisors on Y. If Y is an affine curve, we make the convention that the degree of $\mathfrak{D}$ is equal to the empty set.

It is known that for an affine variety X there is a one-to-one correspondence between $\mathbb{T}$-actions on X and M-gradings on $\mathbb{C}[X]$. The next result (see [AH06, Theorems 3.1, 3.4]) gives a combinatorial description of M-graded algebras corresponding to affine $\mathbb{T}$-varieties of complexity one.

Theorem 2.4.7. *(i) Let $\sigma \subseteq N_{\mathbb{Q}}$ be a strongly convex polyhedral cone and let Y be a smooth curve. If $\mathfrak{D} \in \operatorname{PPDiv}_{\mathbb{Q}}(Y, \sigma)$, then the M-graded subalgebra*

$$A(Y, \mathfrak{D}) := \bigoplus_{m \in \sigma^\vee \cap M} H^0(Y, \mathcal{O}_Y(\lfloor \mathfrak{D}(m) \rfloor)) \otimes \chi^m \subseteq \mathbb{C}(Y) \otimes_{\mathbb{C}} \mathbb{C}[\mathbb{T}],$$

where χ^m is the Laurent monomial corresponding to $m \in M$, defines an affine $\mathbb{T}$-variety $X(\mathfrak{D}) = X(Y, \mathfrak{D})$ of complexity one with rational quotient Y.

(ii) Conversely, if X is an affine $\mathbb{T}$-variety of complexity one, then there exist a strongly convex polyhedral cone $\sigma \subseteq N_{\mathbb{Q}}$, a smooth curve Y, and $\mathfrak{D} \in \operatorname{PPDiv}_{\mathbb{Q}}(Y, \sigma)$ such that the $\mathbb{T}$-variety X is $\mathbb{T}$-isomorphic to $X(\mathfrak{D})$.

We refer to [AH06, Section 8] and [Lan14, Section 4] for the functorial properties and the uniqueness problem of this decomposition.

Now, we explain the combinatorial description of [AHS08] for (non-necessarily affine) $\mathbb{T}$-varieties specialized to the case of the complexity one.

Definition 2.4.8. Let us fix a smooth curve Y. A *divisorial fan* on (Y, N) is a finite set $\mathscr{E} = \{\mathfrak{D}^i \,|\, i \in I\}$ with $\mathfrak{D}^i \in \mathrm{PPDiv}_{\mathbb{Q}}(Y_i, \sigma_i)$, where $Y_i \subseteq Y$ is a Zariski open dense subset and $\sigma_i \subseteq N_{\mathbb{Q}}$ is a strongly convex polyhedral cone, satisfying the following conditions:

(i) For all $i, j \in I$ we have

$$\mathfrak{D}^i \cap \mathfrak{D}^j := \sum_{y \in Y_{ij}} (\mathfrak{D}^i_y \cap \mathfrak{D}^j_y) \cdot [y] \in \mathscr{E},$$

where $Y_{ij} = \{y \in Y_i \cap Y_j \,|\, \mathfrak{D}^i_y \cap \mathfrak{D}^j_y \neq \emptyset\}$.

(ii) For all $i, j \in I$ and for every $y \in Y_{ij}$, the polyhedron $\mathfrak{D}^i_y \cap \mathfrak{D}^j_y$ is a common face of $\mathfrak{D}^i_y$ and $\mathfrak{D}^j_y$.

(iii) The intersection of the degree of $\mathfrak{D}^i$ with the tail σ_{ij} of $\mathfrak{D}^i \cap \mathfrak{D}^j$ is equal to the intersection of the degree of $\mathfrak{D}^j$ with σ_{ij}.

(iv) We have $Y = \bigcup_{i \in I} Y_i$.

By the preceding conditions, the set $\{\sigma_i \,|\, i \in I\}$ generates a fan denoted $\Sigma_\star(\mathscr{E})$. If $\mathscr{E}$ is a divisorial fan on (Y, N), then the natural morphisms (cf [AHS08, Section 5])

$$X(\mathfrak{D}^i) \leftarrow X(\mathfrak{D}^i \cap \mathfrak{D}^j) \rightarrow X(\mathfrak{D}^j)$$

are $\mathbb{T}$-invariant Zariski open immersions. The collection of $\mathbb{T}$-varieties $X(\mathfrak{D}^i)$ can be glued in a $\mathbb{T}$-variety $X(\mathscr{E})$ in which the Zariski open subsets $X(\mathfrak{D}^i \cap \mathfrak{D}^j)$ are identified with the intersections $X(\mathfrak{D}^i) \cap X(\mathfrak{D}^j)$ (see [AHS08, Remark 7.4 (ii)] for the fact that the $\mathbb{C}$-scheme $X(\mathscr{E})$ is separated).

Conversely, if X is a $\mathbb{T}$-variety of complexity one, then there exist a smooth curve Y and a divisorial fan $\mathscr{E}$ on (Y, N) such that the $\mathbb{T}$-variety X is $\mathbb{T}$-isomorphic to $X(\mathscr{E})$ [AHS08, Theorem 5.6]. Note that the proof of this latter fact uses the Sumihiro theorem.

Remark 2.4.9. Certain geometric properties of complexity-one $\mathbb{T}$-varieties can be translated into the language of divisorial fans. For instance, $X(\mathscr{E})$ is a complete variety if and only if Y is a smooth projective curve and

$$\bigcup_{i \in I} \mathfrak{D}^i_y = N_{\mathbb{Q}} \text{ for every } y \in Y$$

(c.f [AHS08, Theorem 7.5]). See also [KKMS73, Chapter II], [LT16, Section 2] for a criterion of smoothness and [PS11, Corollary 3.28] for a criterion of projectivity. Moreover, $X(\mathscr{E})$ is contraction-free if and only if the locus of each element of $\mathscr{E}$ is an affine curve.

Let us fix a divisorial fan $\mathscr{E}$ on (Y, N) such that $X(\mathscr{E})$ is contraction-free. In the next paragraph, we recall the description of the orbits of $\mathbb{T}$ on $X(\mathscr{E})$ in terms of the combinatorial object $\mathscr{E} = \{\mathfrak{D}^i \mid i \in I\}$ (see [AH06, Section 7]).

Denote by $\pi : X(\mathscr{E}) \to Y$ the quotient morphism where the restriction on each open subset $X(\mathfrak{D}^i)$ is given by the inclusion $\mathbb{C}[Y_i] \subseteq A(Y_i, \mathfrak{D}^i)$ (we recall that Y_i is the locus of $\mathfrak{D}^i$). Given $y \in Y$, let us explain how to describe the orbits of the reduced part $\pi^{-1}(y)_{\mathrm{red}}$. For a vertex $v \in \mathfrak{D}^i_y$ consider the cone

$$\lambda(v) = \{m \in \sigma^\vee \mid \forall v' \in \mathfrak{D}^i_y,\ \langle m, v' - v\rangle \geq 0\}.$$

The irreducible components of $\pi^{-1}(y)_{\mathrm{red}} \cap X(\mathfrak{D}^i)$ are identified with the toric varieties $X_{\lambda(v), M_v}$ having weight cone $\lambda(v)$ and weight lattice

$$M_v = \{m \in M \mid \langle m, v\rangle \in \mathbb{Z}\},$$

where v runs the set of vertices of $\mathfrak{D}^i_y$.

Denote by $\mathrm{face}(\mathscr{E})_y$ the set of faces of the polyhedra $\mathfrak{D}^i_y$. We have a bijection

$$F \mapsto O(y, F)$$

between the set $\mathrm{face}(\mathscr{E})_y$ and the set of $\mathbb{T}$-orbits of $\pi^{-1}(y)_{\mathrm{red}}$; the orbit $O(y, F)$ is of dimension $\mathrm{codim}(F) = \mathrm{rank}(N) - \dim(F)$ and is seen geometrically as a common part of the component $X_{\lambda(v), M_v}$ for every vertex v of F.

Let

$$\mathrm{supp}(\mathscr{E}) := \{y \in Y \mid \mathrm{face}(\mathscr{E})_y \neq \Sigma_\star(\mathscr{E})\}.$$

Then we have a natural equivariant identification

$$X(\mathscr{E}) \setminus \pi^{-1}(\mathrm{supp}(\mathscr{E}))_{\mathrm{red}} \simeq (Y \setminus \mathrm{supp}(\mathscr{E})) \times X(\Sigma_\star(\mathscr{E})),$$

where $X(\Sigma_\star(\mathscr{E}))$ is the toric variety for the torus $\mathbb{T}$ associated with the fan $\Sigma_\star(\mathscr{E})$. Indeed, the fibers of the quotient map π over $Y \setminus \mathrm{supp}(\mathscr{E})$ are all isomorphic to $X(\Sigma_\star(\mathscr{E}))$. Since $\mathbb{T}$ is a solvable linear algebraic group, the $\mathbb{T}$-isomorphism above is obtained by the cross section theorem in [Ros56, Section 4, Theorem 10].

In order to understand the topology of $X(\mathscr{E})$, we need the following notion.

Definition 2.4.10. A *prime* $\mathbb{T}$-*cycle* (or a *germ*) of $X(\mathscr{E})$ is a $\mathbb{T}$-stable irreducible reduced Zariski closed subset of $X(\mathscr{E})$.

A combinatorial description of the prime $\mathbb{T}$-cycles of $X(\mathscr{E})$ is given in [Tim11, Section 16.4]. We recall this now.

Let $\mathfrak{D}$ be a σ-polyhedral divisor on a Zariski open dense subset $Y_0 \subseteq Y$. Assuming that Y_0 is affine, we define the set $H(\mathfrak{D})$ as the union of the faces of σ and of the pairs of the form (y, F) where $y \in Y_0$ and F is a face of $\mathfrak{D}_y$.

There exists a bijection between $H(\mathfrak{D})$ and the set of prime $\mathbb{T}$-cycles of $X(\mathfrak{D})$ given as follows. With the same notation as above, let v be in the relative interior of F

where $(y, F) \in H(\mathfrak{D})$ (resp. $F \in H(\mathfrak{D})$ if F is a face of σ). Then we define a discrete valuation $\mathrm{val}_{y,v}$ (resp. val_v) on $A(Y_0, \mathfrak{D})$ via the formula

$$\mathrm{val}_{y,v}(f \otimes \chi^m) = \mathrm{ord}_y(f) + \langle m, v\rangle \text{ resp. } \mathrm{val}_v(f \otimes \chi^m) = \langle m, v\rangle,$$

for every homogeneous element $f \otimes \chi^m \in A(Y_0, \mathfrak{D})$. The associated prime $\mathbb{T}$-cycle is given by the ideal

$$\{\gamma \in A(Y_0, \mathfrak{D}) \setminus \{0\} \,|\, \mathrm{val}_{y,v}(\gamma) > 0 \text{ (resp. } \mathrm{val}_v(\gamma) > 0)\} \cup \{0\}.$$

Let $\mathscr{E}$ be a divisorial fan on (Y, N) with $X(\mathscr{E})$ contraction-free.

Remark 2.4.11. We denote by $H(\mathscr{E})$ the union of all the sets $H(\mathfrak{D})$ where $\mathfrak{D}$ runs through $\mathscr{E}$. Similarly, the set $H(\mathscr{E})$ is in bijection with the set of prime $\mathbb{T}$-cycles of $X(\mathscr{E})$. In the sequel, we will denote by $V(\tau)$ the prime $\mathbb{T}$-cycle associated with $\tau \in H(\mathscr{E})$. Note that in this case every prime $\mathbb{T}$-cycle of $X(\mathscr{E})$ is normal (see [Tim00, Theorem 7]).

Chapter 3

Intersection cohomology for $\mathbb{T}$-varieties of complexity one

This chapter is based on a joint work with Kevin Langlois and most of the results which appear here are included in our paper [AL17]. We give an explicit description (in terms of divisorial fans) of the intersection cohomology Betti numbers of projective contraction-free $\mathbb{T}$-varieties of complexity one. As an intermediate step, we compute the classical Betti numbers for every smooth projective $\mathbb{T}$-variety of complexity one.

All the varieties which appear in this chapter are complex varieties. So, we make the same assumptions as in 2.3.1:

- We suppose the perversity is the middle perversity and we erase it in the notation.
- We consider the complex dimension of X instead of the real one in the definition of intersection cohomology complex.

3.1 Betti numbers of complexity-one smooth $\mathbb{T}$-varieties

The purpose of this section is to give an explicit description of the classical Betti numbers of every smooth projective $\mathbb{T}$-variety of complexity one. Before stating our results, let us introduce some notation.

Let $\mathscr{E}$ be a divisorial fan on (Y, N) such that $X(\mathscr{E})$ is contraction-free. Let $\mathfrak{D} \in \mathscr{E}$ and let $y \in Y$ be a point belonging to the locus of $\mathfrak{D}$. The *associated Cayley cone* of $\mathfrak{D}$ at the point y is the subset $\mathscr{C}(\mathfrak{D})_y$ in $N_{\mathbb{Q}} \oplus \mathbb{Q}$ generated by $(\sigma \times \{0\}) \cup (\mathfrak{D}_y \times \{1\})$. We denote by $\mathscr{C}^-(\mathfrak{D})$ the cone generated by the subset

$$(\sigma \times \{0\}) \cup (\sigma \times \{-1\}) \subseteq N_{\mathbb{Q}} \oplus \mathbb{Q},$$

where σ is the tail of $\mathfrak{D}$. We denote by $\mathscr{E}_y^+$ resp. $\mathscr{E}_y$ the fan generated by

$$\{\mathscr{C}(\mathfrak{D})_y \,|\, \mathfrak{D} \in \mathscr{E} \text{ with } y \text{ in the locus of } \mathfrak{D}\}$$

$$\text{resp. } \{\mathscr{C}^-(\mathfrak{D}), \mathscr{C}(\mathfrak{D})_y \,|\, \mathfrak{D} \in \mathscr{E} \text{ with } y \text{ in the locus of } \mathfrak{D}\}.$$

If $X(\mathscr{E})$ is projective, then $\mathscr{E}_y$ is a complete fan which is a normal fan of a polytope $Q_y(\mathscr{E})$ (compare with [PS11, Corollary 3.28]). The same holds for the fan $\Sigma_\star(\mathscr{E})$; we denote by $Q(\mathscr{E})$ a polytope such that $\Sigma_\star(\mathscr{E})$ is its normal fan.

Let us illustrate the preceding notions by an example where we consider a toric variety with an action of a codimension-one subtorus.

Example 3.1.1. Consider the divisorial fan

$$\mathscr{E} = \{\mathfrak{D}^{i,0}, \mathfrak{D}^{i,\infty} \mid i \in I\}$$

on $(\mathbb{P}^1, N)$, where $\mathfrak{D}^{i,0} \in \mathrm{PPDiv}_{\mathbb{Q}}(\mathbb{P}^1 \setminus \{0\}, \sigma_i)$ and $\mathfrak{D}^{i,\infty} \in \mathrm{PPDiv}_{\mathbb{Q}}(\mathbb{P}^1 \setminus \{\infty\}, \sigma_i)$. Assume that $\mathfrak{D}^{i,0}_y = \mathfrak{D}^{i,\infty}_y = \sigma_i$ for all $i \in I$ and $y \in \mathbb{P}^1 \setminus \{0\}$. Indentifying $\mathbb{P}^1 \setminus \{0\}$ resp. $\mathbb{P}^1 \setminus \{\infty\}$ with $\mathrm{Spec}(\mathbb{C}[t^{-1}])$ resp. $\mathrm{Spec}(\mathbb{C}[t])$ we have the equality

$$A(\mathbb{P}^1 \setminus \{\infty\}, \mathfrak{D}^{i,\infty}) = \mathbb{C}[\mathscr{C}(\mathfrak{D}^{i,\infty})_0^\vee \cap (M \oplus \mathbb{Z})],$$

for every $i \in I$, where the right-hand side is the semigroup algebra of

$$\mathscr{C}(\mathfrak{D}^{i,\infty})_0^\vee \cap (M \oplus \mathbb{Z}).$$

Similarly, we have

$$A(\mathbb{P}^1 \setminus \{0\}, \mathfrak{D}^{i,0}) = \mathbb{C}[\mathscr{C}^-(\mathfrak{D}^{i,0})^\vee \cap (M \oplus \mathbb{Z})].$$

Consequently, $X(\mathscr{E})$ is the toric variety for the torus $\mathbb{T} \times \mathbb{C}^*$ associated with the fan $\mathscr{E}_0$.

The description of the toric variety of Example 3.1.1 allows us to establish relations between h-polynomials of polytopes:

Lemma 3.1.2. *Let $\mathscr{E}$ be a divisorial fan on $(\mathbb{P}^1, N)$ as in Example 3.1.1 and assume that $X(\mathscr{E})$ is smooth projective. Then we have the equality*

$$h(Q_0(\mathscr{E}); t^2) = t^2\, h(Q(\mathscr{E}); t^2) + \sum_{F \in \mathrm{face}(\mathscr{E})_0} (t^2 - 1)^{\mathrm{codim}(F)}.$$

Proof. First of all, by Theorem 2.3.17, we know that $h(Q_0(\mathscr{E}); t^2) = P_{X(\mathscr{E})}(t)$. Let us compute $P_{X(\mathscr{E})}(t)$ by using the method of E-polynomials. Removing the special fiber at the origin of the quotient map $\pi : X(\mathscr{E}) \to \mathbb{P}^1$ gives

$$E(X(\mathscr{E}); u, v) = E(\mathbb{A}^1 \times X(\Sigma_\star(\mathscr{E})); u, v) + E(\pi^{-1}(0)_{\mathrm{red}}; u, v).$$

Using the orbit decomposition of the fibers of the quotient map π we obtain that

$$E(X(\mathscr{E}); u, v) = uv\, E(X(\Sigma_\star(\mathscr{E})); u, v) + \sum_{F \in \mathrm{face}(\mathscr{E})_0} (uv - 1)^{\mathrm{codim}(F)}.$$

The result follows from the substitution $u = v = -t$. □

Corollary 3.1.3. *Let $\mathscr{E}$ be a divisorial fan on (Y, N) such that $X(\mathscr{E})$ is contraction-free. Given $y \in Y$, if the toric variety $X(\mathscr{E}_y)$ associated with the fan $\mathscr{E}_y$ is smooth, then we have the equality.*

$$h(Q_y(\mathscr{E}); t^2) = t^2\, h(Q(\mathscr{E}); t^2) + \sum_{F \in \mathrm{face}(\mathscr{E})_y} (t^2 - 1)^{\mathrm{codim}(F)}.$$

Proof. We can construct a fan $\mathscr{E}'$ on $(\mathbb{P}^1, N)$ as in Example 3.1.1 such that $\Sigma_\star(\mathscr{E}') = \Sigma_\star(\mathscr{E})$ and the fan $\{\mathfrak{D}'_0 | \mathfrak{D}' \in \mathscr{E}'\}$ is equal to $\{\mathfrak{D}_y | \mathfrak{D} \in \mathscr{E}\}$. Then, we have the equalities $Q(\mathscr{E}) = Q(\mathscr{E}')$ and $Q_y(\mathscr{E}) = Q_0(\mathscr{E}')$. So, applying Lemma 3.1.2, we conclude. $\square$

The next result gives, in particular, an explicit formula of the Poincaré polynomial of each smooth projective contraction-free $\mathbb{T}$-variety of complexity one. We consider this result in the general setting of quotient singularities since it will be needed for the proof of Theorem 3.2.8.

Proposition 3.1.4. *Let $X(\mathscr{E})$ be a projective contraction-free $\mathbb{T}$-variety of complexity one corresponding to a divisorial fan $\mathscr{E}$ on (Y, N). Denote by g the genus of the curve Y and by r the cardinality of the finite set* $\mathrm{supp}(\mathscr{E})$*. Assume that every cone of $\mathscr{E}_y$ is generated by a subset of a basis of $N_\mathbb{Q} \oplus \mathbb{Q}$ for all $y \in Y$. Then we have the equality*

$$P_{X(\mathscr{E})}(t) = ((1-r)t^2 + 2gt + 1 - r) h(Q(\mathscr{E}); t^2) + \sum_{y \in \mathrm{supp}(\mathscr{E})} h(Q_y(\mathscr{E}); t^2).$$

Proof. We first compute the Poincaré polynomial $P_{X(\mathscr{E})}(t)$ with the assumption that $X(\mathscr{E})$ is smooth. Using again the method with E-polynomials we have

$$E(X(\mathscr{E}); u, v) = E((Y \setminus \pi^{-1}(\mathrm{supp}(\mathscr{E}))_{\mathrm{red}}) \times X(\Sigma_\star(\mathscr{E})); u, v) + E(\pi^{-1}(\mathrm{supp}(\mathscr{E}))_{\mathrm{red}}; u, v)$$

$$= (uv - (u+v)g + 1 - r) E(X(\Sigma_\star(\mathscr{E})); u, v) + \sum_{y \in \mathrm{supp}(\mathscr{E})} \sum_{F \in \mathrm{face}(\mathscr{E})_y} (uv - 1)^{\mathrm{codim}(F)}.$$

By substituting $u = v = -t$, it follows that

$$P_{X(\mathscr{E})}(t) = \sum_{y \in \mathrm{supp}(\mathscr{E})} \left[t^2 h(Q(\mathscr{E}); t^2) + \sum_{F \in \mathrm{face}(\mathscr{E})_y} (t^2 - 1)^{\mathrm{codim}(F)} \right]$$

$$+((1-r)t^2 + 2gt + 1 - r) h(Q(\mathscr{E}); t^2).$$

According to [KKMS73, Chapter II] the smoothness assumption on $X(\mathscr{E})$ implies that the toric variety $X(\mathscr{E}_y)$ associated with the fan $\mathscr{E}_y$ is smooth for every $y \in Y$. So, applying Corollary 3.1.3, we have the equality

$$h(Q_y(\mathscr{E}); t^2) = t^2\, h(Q(\mathscr{E}); t^2) + \sum_{F \in \mathrm{face}(\mathscr{E})_y} (t^2 - 1)^{\mathrm{codim}(F)}$$

and we obtain the required formula.

In the general case, one can realize $X(\mathscr{E})$ as the quotient $X(\hat{\mathscr{E}})/G$, where $X(\hat{\mathscr{E}})$ is smooth and G is a finite abelian group, by changing the ambient lattice N. Hence, $X(\mathscr{E})$ is rationally smooth [Bri99, Proposition A1 iii)] and so

$$H^j(X(\mathscr{E});\mathbb{Q}) = IH^j(X(\mathscr{E});\mathbb{Q}) \text{ for any } j \in \mathbb{Z}.$$

To obtain the last equalities, we may adapt [HTT08, Proposition 8.2.21] for the intersection cohomology with rational coefficients. Since the quotient map by G induces isomorphisms between rational cohomology groups of $X(\mathscr{E})$ and $X(\hat{\mathscr{E}})$, we conclude that $P_{X(\mathscr{E})}(t) = P_{X(\hat{\mathscr{E}})}(t)$. Finally, we can make the choice to take $Q(\mathscr{E})$ and $Q_y(\mathscr{E})$ so that

$$Q(\mathscr{E}) = Q(\hat{\mathscr{E}}) \text{ and } Q_y(\mathscr{E}) = Q_y(\hat{\mathscr{E}})$$

for every $y \in Y$. This finishes the proof of the proposition. □

Actually, the h-vectors of simplicial polytopes are simple to describe (compare Theorem 2.3.17 with [CMM15, Corollary 3.9]) and the formula of Proposition 3.1.4 can be expressed as the relation

$$P_{X(\mathscr{E})}(t) = ((1-r)t^2 + 2gt + 1 - r)\left(\sum_{i=0}^{d-1} f_i(Q(\mathscr{E}))(t^2-1)^i\right)$$

$$+ \sum_{y\in\operatorname{supp}(\mathscr{E})} \sum_{i=0}^{d-1} f_i(Q_y(\mathscr{E}))(t^2-1)^i,$$

where $d = \dim(X(\mathscr{E}))$ and, for any simplicial polytope Q, we denote by $f_i(Q)$ the number of faces of dimension i.

Now, we complete the computation of Betti numbers of smooth projective $\mathbb{T}$-varieties of complexity one. Note that according to the Luna slice theorem [Lun73, Chapter III] every smooth non-contraction-free $\mathbb{T}$-variety of complexity one is automatically rational (see also [LT16, Section 2]). Hence, we may restrict ourselves to divisorial fans on the projective line.

We want to obtain a formula similar to 3.1.4, where orbits in special and non-special fibers are separated. More precisely, the idea is to use the result [AH06, Theorem 10.1] describing the rational quotient of a non-contraction-free $\mathbb{T}$-variety of complexity one.

Let $\mathscr{E}$ be a divisorial fan on $(\mathbb{P}^1, N)$ such that $X(\mathscr{E})$ is complete. Let us introduce specific notation.

Definition 3.1.5. A polyhedral divisor $\mathfrak{D} \in \mathscr{E}$ with tail τ is *big* if $\tau \cap \deg(\mathfrak{D}) = \emptyset$; it corresponds to a polyhedral divisor verifying that $\mathfrak{D}(u)$ is big (that is, $\deg \mathfrak{D}(u) > 0$) for every u in the relative interior of τ^v.

Remark 3.1.6. Note that an affine $\mathbb{T}$-variety $Z = X(\mathfrak{D})$ of complexity one has its quasi-fan of GIT-chambers (see [AH06, Section 5]) generated by exactly one cone ω_Z, which is the dual of the tail of $\mathfrak{D}$. A proof of this fact can be obtained from results in [Lan14, Section 3]. Let $u \in \omega_Z \cap M$ and consider the morphism of [AH06, Section 10]

$$Y \to Y_u := Proj\left(\bigoplus_{n\in\mathbb{Z}_{\geq 0}} \Gamma(Y, \mathcal{O}(\mathfrak{D}(nu)))\right).$$

If the locus of $\mathfrak{D}$ is a complete curve Y, then Y_u equals Y if $\mathfrak{D}(u)$ is big and Y_u is a point otherwise.

We denote by ${\Sigma_\star}^{\mathrm{big}}(\mathscr{E})$ the subset of elements $\tau \in \Sigma_\star(\mathscr{E})$ with the property that there exists a big polyhedral divisor $\mathfrak{D} \in \mathscr{E}$ with tail τ.

Let us consider the equivariant map $q : \tilde{X} \to X(\mathscr{E})$ obtained by gluing the natural maps

$$\mathrm{Spec}_{Y_0}\left(\bigoplus_{m\in\sigma^\vee\cap M} \mathcal{O}_{Y_0}(\lfloor \mathfrak{D}(m)\rfloor)\otimes\chi^m\right) \to X(\mathfrak{D}),$$

where $\mathfrak{D} \in \mathscr{E}$ and Y_0 resp. σ is the locus resp. the tail of $\mathfrak{D}$.

With the previous notations and using [AH06, Theorem 10.1], the image by q of two orbits in $\tilde{X}$, $O(F_1, y_1)$ and $O(F_2, y_2)$, is the same if and only if $F_1 = \mathfrak{D}^1_{y_1}$ and $F_2 = \mathfrak{D}^2_{y_2}$, where $\mathfrak{D}^1, \mathfrak{D}^2 \in \mathscr{E}$ have the same tail cone τ and τ is not in ${\Sigma_\star}^{\mathrm{big}}(\mathscr{E})$.

We will consider the set

$$\Phi_\star = \{(y, F)\,|y \in \mathrm{supp}(\mathscr{E}) \text{ and } F \in \mathrm{face}(\mathscr{E})_y\}.$$

Moreover, we denote by Φ the quotient set $\Phi_\star/\mathscr{R}$, where the equivalence relation $\mathscr{R}$ is defined as follows. We have $(y, F)\,\mathscr{R}\,(y', F')$ if and only if F and F' have the same tail cone and this tail does not belong to ${\Sigma_\star}^{\mathrm{big}}(\mathscr{E})$. The class of (y, F) modulo $\mathscr{R}$ will be denoted by $[y, F]$. Finally, let us remark that the map

$$\mathrm{codim} : \Phi \to \mathbb{Z}_{\geq 0},\ [y, F] \mapsto \mathrm{codim}(F)$$

is well defined and so for every integer $i \in \mathbb{Z}_{\geq 0}$ we can let

$$\Phi_i = \{[y, F] \in \Phi \,|\, \mathrm{codim}(F) = i\}.$$

Proposition 3.1.7. *Let $\mathscr{E}$ be a divisorial fan on $(\mathbb{P}^1, N)$ such that $X(\mathscr{E})$ is smooth and projective. Let r be the cardinality of* $\mathrm{supp}(\mathscr{E})$*. With the previous notation, we have the equality*

$$P_{X(\mathscr{E})}(t) = (t^2 + 1 - r)\left(\sum_{i=0}^{d-1} c_i(t^2-1)^i\right) + \sum_{i=0}^{d-1} \sharp\Phi_i \cdot (t^2-1)^i,$$

where $d = \dim(X(\mathscr{E}))$ and c_i is the number of cones of ${\Sigma_\star}^{\mathrm{big}}(\mathscr{E})$ of codimension i.

Proof. Let us consider the equivariant map $q : \tilde{X} \to X(\mathscr{E})$.

Let $O(\tau) \subseteq X(\Sigma_\star(\mathscr{E}))$ be the $\mathbb{T}$-orbit corresponding to τ. Note that the set

$$Z_1 = (\mathbb{P}^1 \setminus \mathrm{supp}(\mathscr{E})) \times \bigcup_{\tau \in {\Sigma_\star}^{\mathrm{big}}(\mathscr{E})} O(\tau)$$

identifies with a subset of $\tilde{X}$ by q. Moreover, $q^{-1}(Z_1)$ is the subset of $\tilde{X}$ in which q is an isomorphism. We denote by Z_2 its complement. We have the equality $X(\mathscr{E}) = q(\tilde{X})$ and we observe that $Z_1 \cap q(Z_2) = \emptyset$. Therefore, we have the decomposition

$$E(X(\mathscr{E}); u, v) = E(Z_1; u, v) + E(q(Z_2); u, v),$$

where the second part corresponds to $\sum_{[y,F]\in\Phi} E(O(y, F); u, v)$. Hence by substituting $u = v = -t$, we conclude. $\square$

3.2 Poincaré polynomials in the singular case

3.2.1 A description of the decomposition theorem

In this section, we investigate the decomposition theorem for equivariant proper maps between contraction-free $\mathbb{T}$-varieties of complexity one induced by subdivisions of divisorial fans (see Proposition 3.2.4). Let us introduce some notation.

Let $\mathscr{E}$ be a divisorial fan on (Y, N) such that $X(\mathscr{E})$ is contraction-free. Recall that

$$\pi : X(\mathscr{E}) \to Y$$

stands for the quotient map. We denote by $SH(\mathscr{E})$ the finite subset of $H(\mathscr{E})$ (see Remark 2.4.11) which consists of elements of the form:

(1) $\tau \in \Sigma_\star(\mathscr{E})$ (horizontal type)

(2) pair (y, F) (vertical type) where F is a face of $\mathfrak{D}_y$ for some $\mathfrak{D} \in \mathscr{E}$ and some $y \in \mathrm{supp}(\mathscr{E})$.

The set of elements of type (1) resp. (2) will be denoted by $SH(\mathscr{E})_{\mathrm{hor}}$ resp. $SH(\mathscr{E})_{\mathrm{ver}}$.

Note that the prime $\mathbb{T}$-cycle $V(\tau)$ associated with $\tau \in SH(\mathscr{E})_{\mathrm{hor}}$ is the subvariety obtained as the Zariski closure of $(Y \setminus \mathrm{supp}(\mathscr{E})) \times \tilde{V}(\tau)$ in $X(\mathscr{E})$, where $\tilde{V}(\tau)$ is the orbit closure of $X(\Sigma_\star(\mathscr{E}))$ corresponding to τ and $(Y \setminus \mathrm{supp}(\mathscr{E})) \times X(\Sigma_\star(\mathscr{E}))$ is identified with the complement of $\pi^{-1}(\mathrm{supp}(\mathscr{E}))_{\mathrm{red}}$ in $X(\mathscr{E})$. To every $\tau \in SH(\mathscr{E})_{\mathrm{hor}}$ one can attach a divisorial fan $\mathscr{E}(\tau)$ describing the prime $\mathbb{T}$-cycle $V(\tau)$.

Moreover, every prime $\mathbb{T}$-cycle $V(\tau)$ with $\tau \in SH(\mathscr{E})_{\mathrm{ver}}$ is in $\pi^{-1}(\mathrm{supp}(\mathscr{E}))_{\mathrm{red}}$. If $X(\mathscr{E})$ is projective, then $V(\tau)$ corresponds to a projective toric variety; we denote by Q_τ a polytope such that the fan defining $V(\tau)$ is the normal fan of Q_τ. The notation $\mathscr{E}(\tau)$ resp. Q_τ will apply in Section 3.2.2.

Remark 3.2.1. The prime $\mathbb{T}$-cycles contained in each fiber $\pi^{-1}(y)$ where $y \notin \mathrm{supp}(\mathscr{E})$ correspond to elements of $H(\mathscr{E}) \setminus SH(\mathscr{E})$.

Now, we introduce notions of subdivision of divisorial fans and s-sequences similar to the toric case.

Definition 3.2.2. We say that a divisorial fan $\mathscr{E}'$ on (Y, N) is a *subdivision* of $\mathscr{E}$ if for every $y \in Y$ it verifies:

(i) for all $\mathfrak{D}' \in \mathscr{E}'$, there exists $\mathfrak{D} \in \mathscr{E}$ such that $\mathscr{C}(\mathfrak{D}')_y \subseteq \mathscr{C}(\mathfrak{D})_y$,

(ii) for all $\mathfrak{D} \in \mathscr{E}$, there exist $\mathfrak{D}^1, \ldots, \mathfrak{D}^m \in \mathscr{E}'$ with the same loci as $\mathfrak{D}$ such that $\mathscr{C}(\mathfrak{D})_y = \bigcup_{i\in\{1,\ldots,m\}} \mathscr{C}(\mathfrak{D}^i)_y$.

Clearly a subdivision $\mathscr{E}'$ of $\mathscr{E}$ naturally induces an equivariant birational morphism

$$X(\mathscr{E}') \to X(\mathscr{E})$$

which is proper (see [Tim11, Theorem 12.13]). Note that $X(\mathscr{E}')$ is also contraction-free and that $(\mathscr{E}')^+_y$ is a subdivision fan of $\mathscr{E}^+_y$ for every $y \in Y$. Similarly, $\Sigma_\star(\mathscr{E}')$ is a subdivision of $\Sigma_\star(\mathscr{E})$.

Definition 3.2.3. A *s-sequence* of the subdivision $\mathscr{E}'$ of $\mathscr{E}$ is a sequence $s_{\tau,b}$ of natural numbers where $\tau \in SH(\mathscr{E})$ and $b \in \mathbb{Z}$ such that:

(a) the sequence $s_{b,\tau}$ (for $\tau \in SH(\mathscr{E})_{\text{hor}} = \Sigma_\star(\mathscr{E})$) is a usual s-sequence (see Definition 2.3.16) of the fan subdivision $\Sigma_\star(\mathscr{E}')$ of $\Sigma_\star(\mathscr{E})$,

(b) for every $y \in \operatorname{supp}(\mathscr{E})$, the sequence $s_{b,\tau}$ (for $\tau \in \mathscr{E}^+_y$) is a usual s-sequence of the fan subdivision $(\mathscr{E}')^+_y$ of $\mathscr{E}^+_y$. Here we identify τ with its Cayley cone.

The following is the main result of this section. To prove this result, we observe that a contraction-free $\mathbb{T}$-variety of complexity one is covered by explicit étale open subsets of toric varieties. This idea was developed in [KKMS73, Chapter II] for toroidal embeddings. Thus, we will reduce the proof to the statement 2.3.15.

Proposition 3.2.4. *Let $\mathscr{E}$ be a divisorial fan on (Y, N) corresponding to a contraction-free $\mathbb{T}$-variety $X(\mathscr{E})$ and let $\mathscr{E}'$ be a subdivision of the divisorial fan $\mathscr{E}$. Consider the birational proper equivariant morphism*

$$f : X(\mathscr{E}') \to X(\mathscr{E})$$

given by the subdivision $\mathscr{E}'$. Then, we have an isomorphism

$$f_*\, IC_{X(\mathscr{E}')} \cong \bigoplus_{\tau\in SH(\mathscr{E})} \bigoplus_{b\in\mathbb{Z}} (i_\tau)_* IC_{V(\tau)}^{\oplus s_{\tau,b}}[-b]$$

in the derived category $D^b(X(\mathscr{E}); \mathbb{Q})$, where $s_{\tau,b}$ is an s-sequence of the subdivision $\mathscr{E}'$ and $i_\tau : V(\tau) \to X(\mathscr{E})$ is the inclusion. This implies that, for every Euclidean open subset $U \subseteq X(\mathscr{E})$ and every $j \in \mathbb{Z}$, we have an isomorphism of $\mathbb{Q}$-vector spaces

$$IH^j(f^{-1}(U); \mathbb{Q}) \cong \bigoplus_{\tau\in SH(\mathscr{E})} \bigoplus_{b\in\mathbb{Z}} IH^{j-l_{\tau,b}}(U \cap V(\tau); \mathbb{Q})^{\oplus s_{\tau,b}},$$

where $l_{\tau,b} = b + \dim(X(\mathscr{E})) - \dim(V(\tau))$.

Proof. Let $\mathfrak{D} \in \mathscr{E}$. Denote by Y_0 the locus of $\mathfrak{D}$ and fix a point $z \in Y_0$. Since Y_0 is smooth, there exist two Zariski open neighborhoods

$$z \in U_z \subseteq Y_0 \text{ and } 0 \in V_0 \subseteq \mathbb{A}^1$$

and a surjective étale morphism $\varphi : U_z \to V_0$ sending z to 0 with the condition that $\varphi^{-1}(0) = \{z\}$. Let $\mathfrak{D}^z$ be the polyhedral divisor on $\mathbb{A}^1$ defined by the conditions $\mathfrak{D}^z_0 := \mathfrak{D}_z$ and $\mathfrak{D}^z_y := \sigma$ for any $y \neq 0$, where σ is the tail of $\mathfrak{D}$.

Shrinking U_z and V_0 if necessary, we may suppose that $\mathfrak{D}^z_y = \sigma$ for any $y \in V_0 \setminus \{0\}$ and $\mathfrak{D}_{y'} = \sigma$ for any $y' \in U_z \setminus \{z\}$. Let us denote

$$\mathfrak{D}_{|U_z} := \sum_{y \in U_z} \mathfrak{D}_y \cdot [y] \text{ and } \mathfrak{D}^z_{|V_0} := \sum_{y \in V_0} \mathfrak{D}^z_y \cdot [y].$$

Then the map φ induces an étale morphism $X(\mathfrak{D}_{|U_z}) \to X(\mathfrak{D}^z_{|V_0})$. Indeed, the preceding map may view as the natural projection

$$X(\mathfrak{D}_{|U_z}) \cong X(\mathfrak{D}^z_{|V_0}) \times_{V_0} U_z \to X(\mathfrak{D}^z_{|V_0}).$$

We also remark that $X(\mathfrak{D}^z_{|V_0})$ is a Zariski open subset of the affine toric $\mathbb{T} \times \mathbb{C}^*$-variety $X(\mathscr{C}(\mathfrak{D})_z)$ associated with the cone $\mathscr{C}(\mathfrak{D})_z$ (see Example 3.1.1).

Let $\mathscr{E}'(\mathfrak{D}_{|U_z})$ resp. $\mathscr{E}'(\mathfrak{D}^z_{|V_0})$ be the divisorial fan of polyhedral divisors $\mathfrak{D}'_{|U_z}$ resp. $(\mathfrak{D}')^z_{|V_0}$, where $\mathfrak{D}'$ runs $\mathscr{E}'$ such that $\mathfrak{D}$ and $\mathfrak{D}'$ have the same locus and $\mathscr{C}(\mathfrak{D}')_y \subseteq \mathscr{C}(\mathfrak{D})_y$ for any y belonging to this common locus. Then φ also induces a commutative diagram:

$$\begin{tikzcd}
X(\mathscr{E}'(\mathfrak{D}_{|U_z})) \arrow[r, "f"] \arrow[d, "j"] & X(\mathfrak{D}_{|U_z}) \arrow[d, "i"] \\
X(\mathscr{E}'(\mathfrak{D}^z_{|V_0})) \arrow[r, "f'"] & X(\mathfrak{D}^z_{|V_0})
\end{tikzcd}$$

The horizontal maps f and f' are given by subdivisions of divisorial fans and the vertical maps i and j are étale morphisms. Furthermore, we have the equalities

$$f^{-1}(X(\mathfrak{D}_{|U_z})) = X(\mathscr{E}'(\mathfrak{D}_{|U_z})) \text{ and } (f')^{-1}(X(\mathfrak{D}^z_{|V_0})) = X(\mathscr{E}'(\mathfrak{D}^z_{|V_0})).$$

The map f' is the restriction of a toric map given by a fan subdivision. By Theorem 2.3.15 we have

$$f'_* \, IC_{X(\mathscr{E}'(\mathfrak{D}^z_{|V_0}))} \cong \bigoplus_{\tau \in SH(\mathfrak{D}^z_{|V_0})} \bigoplus_{b \in \mathbb{Z}} (i_\tau)_* IC^{\oplus s_{\tau,b}}_{V'(\tau)}[-b],$$

where $\tau \mapsto V'(\tau)$ is the parametrization of the prime $\mathbb{T}$-cycles in $X(\mathfrak{D}^z_{|V_0})$. Since i, j are smooth of relative dimension 0, we have $i^* = i^!$ and $j^* = j^!$ (see [BBD82, Section 4.2.4]). By construction of the maps i, j, the diagram above is Cartesian. Hence, by base change and the equalities $i^{-1}(V'(\tau)) = V(\tau) \cap X(\mathfrak{D}_{|U_z})$, it follows that

$$f_* IC_{X(\mathscr{E}'(\mathfrak{D}_{|U_z}))} \cong f_* j^* IC_{X(\mathscr{E}'(\mathfrak{D}^z_{|V_0}))} \cong i^* f'_* \, IC_{X(\mathscr{E}'(\mathfrak{D}^z_{|V_0}))} \cong$$

$$\cong \bigoplus_{\tau \in SH(\mathfrak{D}^z_{|V_0})} \bigoplus_{b \in \mathbb{Z}} i^*(i_\tau)_* IC^{\oplus s_{\tau,b}}_{V'(\tau)}[-b].$$

Note that the first isomorphism is given by Lemma 2.3.13. Using the Cartesian diagram

$$\begin{tikzcd} V(\tau) \cap X(\mathfrak{D}_{|U_z}) \arrow[r, "i_\tau"] \arrow[d, "i"] & X(\mathfrak{D}_{|U_z}) \arrow[d, "i"] \\ V'(\tau) \arrow[r, "i_\tau"] & X(\mathfrak{D}^z_{|V_0}) \end{tikzcd}$$

and Lemma 2.3.13, we have, by base change, the isomorphisms

$$(f_*\, IC_{X(\mathscr{E}')})_{|X(\mathfrak{D}_{|U_z})} \cong \bigoplus_{\tau \in SH(\mathscr{E})} \bigoplus_{b \in \mathbb{Z}} (i_\tau)_* IC^{\oplus s_{\tau,b}}_{V(\tau)}[-b]_{|X(\mathfrak{D}_{|U_z})},$$

where $s_{\tau,b}$ is an s-sequence of the divisorial fan subdivision $\mathscr{E}'$. Since $X(\mathscr{E})$ is contraction-free, we can cover $X(\mathscr{E})$ by open subsets

$$X(\mathfrak{D}^1_{|U_{z_1}}), \dots, X(\mathfrak{D}^r_{|U_{z_r}}), \text{ where } \mathfrak{D}^i \in \mathscr{E}, z_i \in Y$$

and the open dense subset $U_{z_i} \subseteq Y$ sastifying $U_{z_i} \cap \operatorname{supp}(\mathfrak{D}) \subseteq \{z_i\}$. By the argument above, we may choose these open subsets so that we have isomorphisms

$$(f_*\, IC_{X(\mathscr{E}')})_{|X(\mathfrak{D}_{|U_{z_i}})} \cong \bigoplus_{\tau \in SH(\mathscr{E})} \bigoplus_{b \in \mathbb{Z}} (i_\tau)_* IC^{\oplus s_{\tau,b}}_{V(\tau)}[-b]_{|X(\mathfrak{D}_{|U_{z_i}})} \text{ for all } i = 1, \dots r. \quad (3.1)$$

Using the uniqueness of the decomposition into simple perverse sheaves [Dim04, Remark 2.5.3] and (3.1), we conclude that

$$f_*\, IC_{X(\mathscr{E}')} \cong \bigoplus_{\tau \in SH(\mathscr{E})} \bigoplus_{b \in \mathbb{Z}} (i_\tau)_* IC^{\oplus s_{\tau,b}}_{V(\tau)}[-b],$$

where $s_{\tau,b}$ is an s-sequence of the divisorial fan subdivision $\mathscr{E}'$. Note that the local systems in the right-hand side of the previous relation are trivial since by (3.1) they are trivial on a Zariski open dense subset (see [Dim04, Remark 2.5.3]). This finishes the proof of the proposition. □

3.2.2 Calculation of the intersection cohomology

The aim of this section is to extend the statement of Proposition 3.1.4 to the singular case. Let us start with some preliminary results.

Lemma 3.2.5. *Let $X(\mathscr{E})$ be a projective contraction-free $\mathbb{T}$-variety of complexity one corresponding to a divisorial fan $\mathscr{E}$ on (Y, N). Then, there exists a subdivision $\mathscr{E}'$ of the divisorial fan $\mathscr{E}$ such that, for any $y \in Y$, each cone of $\mathscr{E}'_y$ is generated by a subset of a basis of $N_{\mathbb{Q}} \oplus \mathbb{Q}$ and the natural morphism $X(\mathscr{E}') \to X(\mathscr{E})$ is projective.*

Proof. First of all, we consider a star subdivision (see [CLS11, Section 11.1]) for every one-dimensional cone of $\Sigma_\star(\mathscr{E})$. This induces a subdivision $\mathscr{E}''$ of $\mathscr{E}$. Secondly, to each $y \in \mathrm{supp}(\mathscr{E}'')$, we consider a star subdivision of $(\mathscr{E}'')_y^+$ for every one-dimensional cone of $(\mathscr{E}'')_y^+ \setminus \Sigma_\star(\mathscr{E}'')$. This last process induces again a subdivision $\mathscr{E}'$ of $\mathscr{E}$ (see [LPR16, Section 5.3] for more details). We conclude, by [PS11, Corollary 3.28] and the proof of [CLS11, Proposition 11.1.7], that we may find the subdivision $\mathscr{E}'$ as required. $\square$

As in Section 3.1, the next result establishes relations between h-polynomials of polytopes starting with Example 3.1.1.

Lemma 3.2.6. *Let $\mathscr{E}, \mathscr{E}'$ be divisorial fans on $(\mathbb{P}^1, N)$ as in Example 3.1.1. Assume that $\mathscr{E}_0'$ is a subdivision of $\mathscr{E}_0$ such that $X(\mathscr{E}')$ is projective and each cone of $\mathscr{E}_0'$ is generated by a subset of a basis of $N_\mathbb{Q} \oplus \mathbb{Q}$. Let $d = \dim(X(\mathscr{E}))$ and $d_\tau = \dim(V(\tau))$ for $\tau \in H(\mathscr{E})$. Then, we have the formula*

$$h_{j+d}(Q_0(\mathscr{E})) = h_{j+d}(Q_0(\mathscr{E}')) - \sum_{\substack{\tau \in SH(\mathscr{E})_{\mathrm{hor}} \\ \tau \neq 0}} \sum_{b \in \mathbb{Z}} s_{\tau,b} \cdot h_{j+d_\tau-b}(Q_0(\mathscr{E}(\tau)))$$

$$- \sum_{\substack{\tau \in SH(\mathscr{E})_{\mathrm{ver}} \\ \tau \in \mathscr{E}_0^+}} \sum_{b \in \mathbb{Z}} s_{\tau,b} \cdot h_{j+d_\tau-b}(Q_\tau)$$

for every $j \in \mathbb{Z}_{\geq 0}$, where $s_{\tau,b}$ is any s-sequence of the subdivision $\mathscr{E}'$.

Proof. Applying Theorem 2.3.15 for the (toric) proper map $X(\mathscr{E}') \to X(\mathscr{E})$ induced by the subdivision $\mathscr{E}_0'$ of $\mathscr{E}_0$ we obtain

$$b_{j+d}(X(\mathscr{E}')) = \sum_{\tau \in SH(\mathscr{E})} \sum_{b \in \mathbb{Z}} s_{\tau,b} \cdot b_{j+d_\tau-b}(X(\mathscr{E}(\tau)))$$

for every $j \in \mathbb{Z}$. We conclude by using Theorem 2.3.17 and the fact that $s_{0,0} = 1$ and $s_{0,b} = 0$ for all $b \in \mathbb{Z} \setminus \{0\}$. $\square$

The reader may remark that the formula of the next result depends only of the tail fans $\Sigma_\star(\mathscr{E})$ and $\Sigma_\star(\mathscr{E}')$. Furthermore, the proof of this result is a straightforward consequence of Theorem 2.3.15 and so, we omitted it.

Lemma 3.2.7. *Let $\mathscr{E}$ be a divisorial fan on (Y, N) such that $X(\mathscr{E})$ is contraction-free and projective. Let $\mathscr{E}'$ be a subdivision of $\mathscr{E}$ as in Lemma 3.2.5. We recall that $Q(\mathscr{E})$ denotes a polytope such that $\Sigma_\star(\mathscr{E})$ is a normal fan of $Q(\mathscr{E})$. Then, we have the equality*

$$h_{j+d}(Q(\mathscr{E})) = h_{j+d}(Q(\mathscr{E}')) - \sum_{\substack{\tau \in SH(\mathscr{E})_{\mathrm{hor}} \\ \tau \neq 0}} \sum_{b \in \mathbb{Z}} s_{\tau,b} \cdot h_{j+d_\tau-b}(Q(\mathscr{E}(\tau)))$$

for every $j \in \mathbb{Z}$, where $s_{\tau,b}$ is any s-sequence of the subdivision $\mathscr{E}'$.

The following is our main result.

Theorem 3.2.8. *Let $X(\mathscr{E})$ be a singular projective contraction-free $\mathbb{T}$-variety of complexity one corresponding to a divisorial fan $\mathscr{E}$ on (Y,N). Denote by g the genus of the curve Y and by r the cardinal of the set $\operatorname{supp}(\mathscr{E})$. Then, we have the equality*

$$P_{X(\mathscr{E})}(t) = ((1-r)t^2 + 2gt + 1 - r)h(Q(\mathscr{E});t^2) + \sum_{y\in\operatorname{supp}(\mathscr{E})} h(Q_y(\mathscr{E});t^2).$$

In particular, if $X(\mathscr{E})$ is rational, then $IH^{2j+1}(X(\mathscr{E});\mathbb{Q}) = 0$ for every $j \in \mathbb{Z}$.

Proof. We show the result by induction on the dimension d of $X(\mathscr{E})$. In the initial step $d = 1$, we have $X(\mathscr{E}) = Y$. So, $P_{X(\mathscr{E})}(t) = t^2 + 2gt + 1$ and the result holds in this step. Assume that the result holds in dimension $< d$ where $d \geq 2$. Let us consider a subdivision $\mathscr{E}'$ of $\mathscr{E}$ as in Lemma 3.2.5. Using Proposition 3.2.4, we can write for every $j \in \mathbb{Z}$:

$$b_{j+d}(X(\mathscr{E})) = b_{j+d}(X(\mathscr{E}')) - \sum_{\substack{\tau\in SH(\mathscr{E})\\ \tau\neq 0}} \sum_{b\in\mathbb{Z}} s_{\tau,b}\cdot b_{j+d_\tau-b}(X(\mathscr{E}(\tau)))$$

$$= b_{j+d}(X(\mathscr{E}')) - \sum_{\substack{\tau\in SH(\mathscr{E})_{\mathrm{hor}}\\ \tau\neq 0}} \sum_{b\in\mathbb{Z}} s_{\tau,b}\cdot b_{j+d_\tau-b}(X(\mathscr{E}(\tau)))$$

$$- \sum_{y\in\operatorname{supp}(\mathscr{E})} \sum_{\substack{\tau\in SH(\mathscr{E})_{\mathrm{ver}}\\ \tau\in\mathscr{E}_y^+}} \sum_{b\in\mathbb{Z}} s_{\tau,b}\cdot b_{i+d_\tau-b}(X(\mathscr{E}(\tau))).$$

We apply the induction process for the prime $\mathbb{T}$-cycles $V(\tau) = X(\mathscr{E}(\tau)) \subsetneq X(\mathscr{E})$ so that

$$b_{j+d}(X(\mathscr{E})) = \sum_{y\in\operatorname{supp}(\mathscr{E})} h_{j+d}(Q_y(\mathscr{E}'))$$

$$+(1-r)\,h_{j+d-2}(Q(\mathscr{E}')) + 2g\,h_{j+d-1}(Q(\mathscr{E}')) + (1-r)\,h_{j+d}(Q(\mathscr{E}'))$$

$$- \sum_{\substack{\tau\in SH(\mathscr{E})_{\mathrm{hor}}\\ \tau\neq 0}} \sum_{b\in\mathbb{Z}} s_{\tau,b}\cdot [\sum_{y\in\operatorname{supp}(\mathscr{E})} h_{j+d_\tau-b}(Q_y(\mathscr{E}(\tau)))$$

$$+(1-r)\,h_{j+d_\tau-b-2}(Q(\mathscr{E}(\tau))) + 2g\,h_{j+d_\tau-b-1}(Q(\mathscr{E}(\tau))) + (1-r)\,h_{j+d_\tau-b}(Q(\mathscr{E}(\tau)))]$$

$$- \sum_{y\in\operatorname{supp}(\mathscr{E})} \sum_{\substack{\tau\in SH(\mathscr{E})_{\mathrm{ver}}\\ \tau\in\mathscr{E}_y^+}} \sum_{b\in\mathbb{Z}} s_{\tau,b}\cdot h_{j+d_\tau-b}(Q_\tau).$$

We observe that, since $\operatorname{supp}(\mathscr{E}(\tau)) \subseteq \operatorname{supp}(\mathscr{E})$ for any $\tau \in SH(\mathscr{E})$, we may substitute the sets $\operatorname{supp}(\mathscr{E}(\tau))$ by the set $\operatorname{supp}(\mathscr{E})$ in the second and third sums of the right-hand side of the preceding equality. Using Lemmas 3.2.6 and 3.2.7, we have

$$b_{j+d}(X(\mathscr{E})) = (1-r)\cdot[h_{j+d-2}(Q(\mathscr{E}')) - \sum_{\substack{\tau\in SH(\mathscr{E})_{\mathrm{hor}}\\ \tau\neq 0}} \sum_{b\in\mathbb{Z}} s_{\tau,b}\cdot h_{j+d_\tau-b-2}(Q(\mathscr{E}(\tau)))]$$

$$+2g \cdot [h_{j+d-1}(Q(\mathscr{E}')) - \sum_{\substack{\tau \in SH(\mathscr{E})_{\mathrm{hor}} \\ \tau \neq 0}} \sum_{b \in \mathbb{Z}} s_{\tau,b} \cdot h_{j+d_\tau-b-1}(Q(\mathscr{E}(\tau)))]$$

$$+(1-r) \cdot [h_{j+d}(Q(\mathscr{E}')) - \sum_{\substack{\tau \in SH(\mathscr{E})_{\mathrm{hor}} \\ \tau \neq 0}} \sum_{b \in \mathbb{Z}} s_{\tau,b} \cdot h_{j+d_\tau-b}(Q(\mathscr{E}(\tau)))]$$

$$+ \sum_{y \in \mathrm{supp}(\mathscr{E})} [h_{j+d}(Q_y(\mathscr{E}')) - \sum_{\substack{\tau \in SH(\mathscr{E}) \\ \tau \neq 0}} \sum_{b \in \mathbb{Z}} s_{\tau,b} \cdot h_{j+d_\tau-b}(Q_y(\mathscr{E}(\tau)))$$

$$- \sum_{\substack{\tau \in SH(\mathscr{E})_{\mathrm{ver}} \\ \tau \in \mathscr{E}_y^+}} \sum_{b \in \mathbb{Z}} s_{\tau,b} \cdot h_{j+d_\tau-b}(Q_\tau)]$$

$$= (1-r)h_{j+d-2}(Q(\mathscr{E})) + 2g h_{j+d-1}(Q(\mathscr{E})) + (1-r)h_{j+d}(Q(\mathscr{E})) + \sum_{y \in \mathrm{supp}(\mathscr{E})} h_{j+d}(Q_y(\mathscr{E})).$$

This gives the formula for the Poincaré polynomial $P_{X(\mathscr{E})}(t)$. Finally, for the last claim we apply the formula for $g = 0$. $\square$

Chapter 4

Intersection space cohomologically constructible complexes

This chapter is based on a joint work with Javier Fernández de Bobadilla. We study the Banagl construction of intersection spaces. Given a topological pseudomanifold, we give a method to construct, if it is possible, a pair of spaces which generalizes Banagl construction (the intersection space pair). Moreover, we define a class of constructible complexes (the intersection space complexes) such that, if the intersection space pair exist, this class is non-empty and the hypercohomology of some of the complexes in it equals the relative cohomology of the intersection space pair. This allows us to give some examples of topological pseudomanifolds for which Banagl construction does not generalize. Finally, we prove some duality properties of intersection space complexes.

4.1 Main results

Since some parts of this chapter are very technical, this section pretends to summarize the principal results and ideas to guide the reader along its lecture. We describe the content of the chapter section by section. The reader may jump to the corresponding sections for a full exposition.

4.1.1 Topological constructions

The chapter starts with a topological construction of pairs of intersection spaces in Section 4.2.

Banagl construction of the intersection space [Ban10] for a d-dimensional topological pseudomanifold X with isolated singularities for a given perversity $\bar{p}$ runs as follows. Let $\Sigma = \{p_1, ..., p_r\}$ be the singular set. Around p_i consider a conical neighbourhood B_i. Let L_i denote the link ∂B_i. Let $\bar{q}$ be the complementary perversity of $\bar{p}$. Consider a homological truncation

$$(L_i)_{\leq \bar{q}(d)} \to L_i.$$

This is a mapping of spaces inducing isomorphism in homology in degrees up to $\bar{q}(d)$ and such that $H_i((L_i)_{\leq \bar{q}(m)})$ vanish for $i > \bar{q}(d)$. Assume for simplicity that the truncation map is an inclusion. Construct the intersection space replacing each of the B_i's by the cone over $(L_i)_{\leq \bar{q}(d)}$, call the resulting space Z. The vertices of the cone are called $\Sigma = \{p_1, ..., p_r\}$ as well. The intersection space is the result of attaching the cone over Σ to Z. If the truncation map is not an inclusion, one may force this using an appropriate homotopy model for it. The intersection space homology is the reduced homology of the intersection space.

The construction for the case of topological pseudomanifolds $X = X_d \supset X_{d-m}$ of dimension d with a single singular stratum of codimension m contained in [Ban10] and [BaCh] is the following generalization. Let T be a tubular neighbourhood of the singular set X_{d-m}. Consider the locally trivial fibration $T \to X_{d-m}$, and let $\partial T \to X_{d-m}$ be the associated fibration of links. Consider a fibrewise homology truncation

$$\partial T_{\leq \bar{q}(m)} \to \partial T.$$

This is a morphism of locally trivial fibrations which is a $\bar{q}(m)$ homology truncation at each fibre. Remove T from X and replace it by the fibrewise cone over $\partial T_{\leq \bar{q}(m)}$ (see Definition 4.2.3); call the resulting space Z. As before Σ is a subspace of Z. The intersection space is the result of attaching the cone over Σ to Z and the intersection space homology is the reduced homology of the intersection space.

Notice that the intersection space homology coincides with the relative homology $H_*(Z, \Sigma)$. This observation is the starting point of our construction for more than two strata and of our constructible complex approach to intersection space homology. Consider a pseudomanifold with stratification

$$X = X_d \supset X_{d-2} \supset ... \supset X_0 \supset X_{-1} = \emptyset$$

and a suitable system of tubular neighbourhoods of the strata (the conical structure of Definition 4.2.4). A great variety of topological pseudomanifolds can be endowed with this structure (see Remark 4.2.10).

Our topological construction of intersection spaces consist in modifying the space X inductively, going each step deeper in the codimension of the strata, by taking successive fibrewise homology truncations. In doing so, one necessarily modifies the singular set X_{d-2} and, as a result, the singular set is not going to be contained in the modified space Z. Instead, one obtains a modification Y of X_{d-2} contained in Z. One obtains a pair (Z, Y), and it is the homology of this pair our definition of intersection space homology. An important feature of the construction is that one needs to adopt the view point of pairs of spaces right from the beginning if one wants to have a chance of proving duality results; this incarnates in the need of taking homology truncations of pairs of spaces. This new feature appears from the depth 2 strata and hence it did not appear in Banagl constructions explained above. It also may happen that, if the singular stratum, is of small dimension in comparison with the perversity, the situation does not appear at all.

As in Banagl construction, the homology truncations need not be inclusions. This forces us to work with an adequate homotopy model for X.

It is important to record for future reference that at the k-th inductive Step of the construction one obtains a pair of spaces $(I^{\bar{p}}_k X, I^{\bar{p}}_k(X_{d-2}))$ which contain X_{d-k-1} and verify

1. the pair $(I^{\bar{p}}_k X \setminus X_{d-k-1}, I^{\bar{p}}_k(X_{d-2}) \setminus X_{d-k-1})$ is an intersection space pair of $X \setminus X_{d-k-1}$.
2. there is a system of tubular neighbourhoods T of $X_{d-k-1} \setminus X_{d-k-2}$ in $I^{\bar{p}}_k X \setminus X_{d-k-2}$ such that we have locally trivial fibrations of pairs
$$T \cap (I^{\bar{p}}_k X \setminus X_{d-k-2}, I^{\bar{p}}_k(X_{d-2}) \setminus X_{d-k-2}) \to X_{d-k-1} \setminus X_{d-k-2},$$
$$\partial T \cap (I^{\bar{p}}_k X \setminus X_{d-k-2}, I^{\bar{p}}_k(X_{d-2}) \setminus X_{d-k-2}) \to X_{d-k-1} \setminus X_{d-k-2},$$
being the first the fibrewise cone over the second (see Definition 4.2.3).

The second locally trivial fibration is called *the fibration of link pairs at the k-th step of the construction.*

The construction follows the scheme of obstruction theory: it is inductive and, at each step, choices are made. The next step may be obstructed and this may depend on the previous choices. The obstruction consists in the impossibility of constructing a fibrewise homology truncation of the fibration of link pairs at the k-th step of the construction.

When there is a set of choices so that the process terminates, we say that an intersection space pair exists. It needs not be unique. In the case where the conical structure is trivial (see Definition 4.2.9), the intersection space pair exists (see Corollary 4.2.26).

4.1.2 From topology to constructible complexes

In the rest of the chapter, we investigate the existence and uniqueness of intersection space pairs and their duality properties by sheaf theoretic methods. For this, we associate to each intersection space pair an element in the derived category of constructible complexes whose hypercohomology computes the rational cohomology of the intersection space pair.

To get this, we need to construct a sequence of intersection space pairs which modify the pair (X_d, X_{d-2}) in increasingly smaller neighbourhoods of the strata of X. This is done in Section 4.3.

In Section 4.4, we exploit the sequence of intersection spaces to derive a constructible complex IS (see Definition 4.4.15) and prove, in Theorem 4.4.16, that the hypercohomology of IS recovers the cohomology of the intersection space pair. Finally, in Theorem 4.4.18, we prove that IS satisfies a set of properties in the same spirit that those that characterize intersection cohomology complexes in [GM83]. This is the basis for the axiomatic treatment of the next section.

4.1.3 A constructible complex approach to intersection space (co)homology

In Section 4.5, we take an axiomatic approach to intersection space complexes in the same way as Goreski- McPherson approach to intersection cohomology in [GM83, section 3.3]. We define two sets of properties in the derived category of cohomologically constructible sheaves on X. The first set are the properties of the intersection cohomology sheaf composed with a shift. The second set of properties are inspired by Theorem 4.4.18.

We will call a complex of sheaves verifying the second set of properties intersection space complex of X (Definition 4.5.2). Theorem 4.4.18 implies that if there exist an intersection space pair of X (see Definition 4.2.22), then there exist an intersection space complex of X whose hypercohomology coincides with the cohomology of the intersection space pair. Moreover, we compare the support and cosupport properties of intersection cohomology complexes and intersection space complexes. From the comparison, one sees that intersection space complexes, except possibly in the case of isolated singularities, can not be perverse sheaves.

At this point, we investigate in a purely sheaf theoretical way in which conditions an intersection space complex may exist. For $k = 2, ..., d$, we define $U_k := X \setminus X_{d-k}$ and we denote the canonical inclusions by $i_k : U_k \to U_{k+1}$ and $j_k : X_{d-k} \setminus X_{d-(k+1)} \to U_{k+1}$.

In Theorem 4.6.3, we give necessary and sufficient conditions for the existence of intersection space complexes. As in the topological setting, the construction proceeds inductively on each time deeper strata. At the $k+1$-th step, we have constructed an intersection space complex IS_k on $X \setminus X_{d-k-1}$ such that the complex $j^*_{k+1} i_{k+1*} IS_k$ on $X_{d-k-1} \setminus X_{d-k-2}$ is cohomologically constructible. Comparing with the topological construction these local systems are the cohomology local systems of the fibration of link pairs at the $k+1$-th step of the construction. One can consider the natural triangle in the derived category:

$$\tau_{\leq \bar{q}(k+1)} j^*_{k+1} i_{k+1*} IS_k \to j^*_{k+1} i_{k+1*} IS_k \to \tau_{>q(k-1)} j^*_{k+1} i_{k+1*} IS_k \xrightarrow{[1]} .$$

The obstruction to perform the next step in the construction is the obstruction to split the triangle in the derived category. This is the constructible sheaf counterpart of the possibility to construct fibrewise homology truncations in the topological world. In Theorem 4.6.3, we study the parameter spaces classifying the possible intersection space complexes in step $k+1$ having fixed the step k (they are not unique in general). In the same Theorem, for the sake of comparison we provide a proof of the existence and uniqueness of intersection cohomology complexes using the same kind of techniques.

There are extension groups controlling the obstructions for existence and uniqueness at each step. They are recorded in Corollary 4.6.6.

4.1.4 Classes of spaces admitting intersection space complexes and counterexamples

From the previous section, it is clear that the spaces admitting intersection space complexes need to be special. In this Section 4.7, we find two sufficient conditions for this, yielding an ample class of (yet special) examples:

1. The link fibrations are cohomologically trivial in the sense of Definition 4.7.1. This includes cases studied by Banagl in [Ban10] and the important case of toric varieties (Example 4.7.6).
2. The strata are of "cohomological dimension at most 1 for local systems" in the sense of Definition 4.7.7. This includes the case studied by Banagl in [Ban12] and the case of algebraic varieties with critical set of dimension 1 which are sufficiently singular, in the sense that there are no positive dimensional compact strata (Corollary 4.7.9).

From our previous results, it is clear that a necessary condition for the existence of an intersection space is the existence of an intersection space complex. We find a few examples (see Example 4.7.12, Example 4.7.13 and Example 4.7.14) not admitting intersection space complex. The last example is an algebraic variety with two strata and the perversity is the middle one. So, one should not expect that algebraicity helps in the existence of intersection spaces. The idea to produce the examples is to observe that, if an space admits intersection space complexes, certain differentials in the Leray spectral sequence of the fibrations of links have to vanish (Proposition 4.7.10 and Corollary 4.7.11).

4.1.5 Duality

In Section 4.8, we prove that, if $\bar{p}$ and $\bar{q}$ are complementary perversities and $IS_{\bar{p}}$ is an intersection space complex for perversity $\bar{p}$, then its Verdier dual is an intersection space complex for perversity $\bar{q}$ (Theorem 4.8.1). So, the Verdier duality functor exchanges the sets of intersection space complexes for complementary perversities. The proof follows the axiomatic treatment of [GM83] for intersection cohomology complexes.

A surprising consequence is that the existence of intersection space complexes is equivalent for complementary perversities (Corollary 4.8.2).

Then, we move to the case of depth 1 stratifications and prove, in Proposition 4.8.3, that, for generic choices of the intersection space complexes, the Betti numbers are always the same (they are minimal). Then, in Theorem 4.8.6, we show that the Betti numbers symmetry predicted by Poincaré Duality for complementary perversities is satisfied for generic intersection space Betti numbers.

4.1.6 Open questions

Here is a list of natural questions for further study:

(1) Assume that the intersection space complex exist. Does there exist an associated rational homotopy intersection space? Are there further restrictions than the existence of the intersection space complex?

(2) If the intersection space complex exist, can one define on its hypercohomology a natural internal cup product? Can one find a product turning its space of sections into a differential graded algebra inducing a cup product? If the intersection space exists, the answer is positive.

(3) If X is an algebraic variety, are intersection space complexes for the middle perversity self-dual with respect to Verdier duality?

4.2 A topological construction of Intersection spaces

4.2.1 Topological preliminaries

First, we give some basic definitions about t-uples of spaces in order to fix notation.

Definition 4.2.1. (1) A t-uple of spaces is a set of topological spaces $(Z_1, ..., Z_t)$.

(2) A morphism from a t-uple of spaces into a space $(Z_1, ..., Z_t) \to Z$ is a set of morphisms $\varphi_i : Z_1 \to Z$.

(3) A morphism between t-uples of spaces $(Z_1, ..., Z_t) \to (Z_1', ..., Z_t')$ is a set of morphisms $\varphi_i : Z_i \to Z_i'$.

(4) The mapping cylinder of a morphism $\varphi = (\varphi_1, ..., \varphi_t) : (Z_1, ..., Z_t) \to Z$, $cyl(\varphi)$, is the union of the t-uple $(Z_1, ..., Z_t) \times [0,1]$ with $(Z, \mathrm{Im}(\varphi_2), ..., \mathrm{Im}(\varphi_t))$ with the equivalence relation $\sim$ such that for $i = 1, ..., t$ and for every $x \in Z_i$, we have $(x, 1) \sim \varphi_i(x)$.

Remark 4.2.2. Remember that the mapping cylinder of a morphism of spaces $f : X \to Y$ is $cyl(f) := (X \times [0,1] \sqcup Y)/\sim$ where $\sim$ is the equivalence relation such that for every $x \in X$, $(x, 1) \sim f(x)$.

Definition 4.2.3. Let $\sigma : (Z_1, ..., Z_t) \to B$ be a locally trivial fibration of t-uples of spaces. The *cone* of σ over the base B is the locally trivial fibration

$$\pi : cyl(\sigma) \to B,$$

where $cyl(\sigma)$ is the mapping cylinder of σ, $\pi(x,t) := \sigma(x)$ for $(x,t) \in (Z_1, ..., Z_t) \times [0,1]$ and $\pi(b) := b$ for $b \in B$ (the definition of π is compatible with the identifications made to construct $cyl(\sigma)$). The cone over a fibration has a canonical *vertex section*

$$s : B \to cyl(\sigma)$$

sending any $b \in B$ to the vertex of the cone $(cyl(\sigma))_b$.

The following figure shows $cyl(\sigma)$ when the fibre of σ is the pair $(\mathbb{T}, \Sigma)$ where $\mathbb{T}$ is a torus and Σ is isomorphic to S^1.

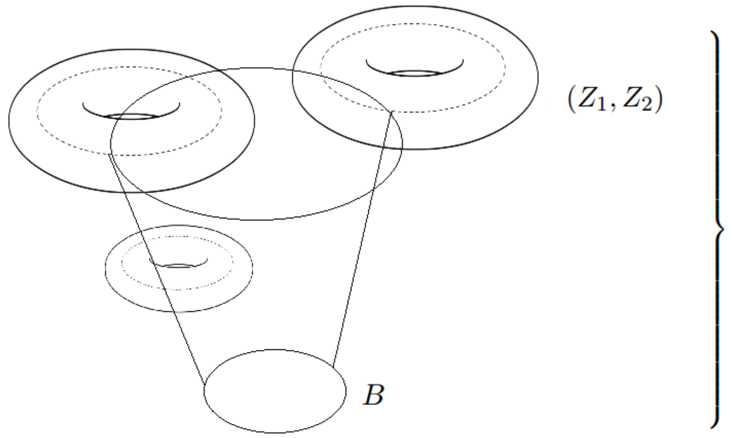

Figure 4.1

The following new notion is important in our constructions.

Definition 4.2.4. Let (X, Y) be a pair of topological spaces and let

$$X_{d-k} \supset X_{d-k-1} \supset ... \supset X_0 \supset X_{-1} = \emptyset$$

be a topological pseudomanifold such that X_{d-k} is a subspace of Y. We say that the pair (X, Y) has a *conical structure with respect to the stratified subspace* if for every $r \geq k$ there exits an open neighbourhood TX_{d-r} of $X_{d-r} \setminus X_{d-r-1}$ in $X \setminus X_{d-r-1}$, with the following properties:

(1) Let $\overline{TX_{d-r}}$ be the closure of TX_{d-r} in X. There is a locally trivial fibration of $2(r-k+1)$-uples of spaces

$$\begin{array}{c}(\overline{TX_{d-r}} \setminus X_{d-r-1}) \cap (X, Y, \overline{TX_{d-k}}, X_{d-k}, \overline{TX_{d-k-1}}, X_{d-k-1}, ..., \overline{TX_{d-r+1}}, X_{d-r+1}) \\ \Big\downarrow \sigma_{d-r} \\ X_{d-r} \setminus X_{d-r-1}\end{array}$$

such that its restriction to the boundary

$$\begin{array}{c}(\partial\overline{TX_{d-r}} \setminus X_{d-r-1}) \cap (X, Y, \overline{TX_{d-k}}, X_{d-k}, \overline{TX_{d-k-1}}, X_{d-k-1}, ..., \overline{TX_{d-r+1}}, X_{d-r+1}) \\ \Big\downarrow \sigma^{\partial}_{d-r} \\ X_{d-r} \setminus X_{d-r-1}\end{array}$$

is a locally trivial fibration.

(2) The fibration σ_{d-r} is the cone of σ^{∂}_{d-r} over the base $X_{d-r} \setminus X_{d-r-1}$.

(3) Let $k \leq r_1 < r_2 \leq d$ and consider the isomorphism induced by property (4.2.4)

$$\overline{TX_{d-r_1}} \cap (\overline{TX_{d-r_2}} \setminus X_{d-r_2-1}) \cong (\overline{TX_{d-r_1}} \cap (\partial\overline{TX_{d-r_2}} \setminus X_{d-r_2-1})) \times [0,1]/\sim$$

where $\sim$ is the equivalence relation of the mapping cylinder.

If we remove X_{d-r_1-1} in both parts of this isomorphism, we obtain an isomorphism

$$\phi_{r_1,r_2} : (\overline{TX_{d-r_1}} \setminus X_{d-r_1-1}) \cap \overline{TX_{d-r_2}} \cong ((\overline{TX_{d-r_1}} \setminus X_{d-r_1-1}) \cap \partial\overline{TX_{d-r_2}}) \times [0,1).$$

Note that since X_{d-r_2} is contained in X_{d-r_1-1}, the vertex section of the cone is not included in the previous spaces.

With this notation, we have the equality

$$(\sigma_{d-r_1})_{|(\overline{TX_{d-r_1}}\setminus X_{d-r_1-1})\cap\overline{TX_{d-r_2}}} =$$
$$= \phi^{-1}_{r_1,r_2} \circ ((\sigma_{d-r_1})_{|(\overline{TX_{d-r_1}}\setminus X_{d-r_1-1})\cap\partial\overline{TX_{d-r_2}}}, Id_{[0,1)}) \circ \phi_{r_1,r_2},$$

that is, the fibration σ_{d-r_1} in the intersection $(\overline{TX_{d-r_1}} \setminus X_{d-r_1-1}) \cap \overline{TX_{d-r_2}}$ is determined by its restriction to $(\overline{TX_{d-r_1}} \setminus X_{d-r_1-1}) \cap \partial\overline{TX_{d-r_2}}$.

(4) Let $k \leq r_1 < r_2 \leq d$. If $\partial\overline{TX_{d-r_2}} \cap (X_{d-r_1} \setminus X_{d-r_1-1}) \neq \emptyset$, then we have the following equality of $2(r-k+1)$-uples

$$(\overline{TX_{d-r_1}}\setminus X_{d-r_1-1})\cap(X, Y, \overline{TX_{d-k}}, X_{d-k}, \overline{TX_{d-k-1}}, X_{d-k-1}, ..., \overline{TX_{d-r+1}}, X_{d-r+1})\cap$$
$$\cap\partial\overline{TX_{d-r_2}} = \sigma^{-1}_{d-r_1}(\partial\overline{TX_{d-r_2}} \cap (X_{d-r_1} \setminus X_{d-r_1-1}))$$

and, in this space, we have

$$\sigma^{\partial}_{d-r_2} \circ \sigma_{d-r_1} = \sigma^{\partial}_{d-r_2}.$$

Remark 4.2.5. The fibration σ^{∂}_{d-r} is the fibration of links of $X_{d-r} \setminus X_{d-r-1}$ and the fibration σ_{d-r} is the fibration associated to a tubular neighborhood.

The fact that these fibrations of $(r-k+2)$-uples are locally trivial yields that the intersection of the link, L^x, of the point $x \in X_{d-r} \setminus X_{d-r-1}$ with the open neighbourhoods $TX_{d-k}, TX_{d-k-1}, ..., TX_{d-r+1}$ only depends on the connected component of $X_{d-r} \setminus X_{d-r-1}$ containing x.

Remark 4.2.6. Using properties (3) and (4) of Definition 4.2.4, we can also deduce that

$$(\overline{TX_{d-r_1}}\setminus X_{d-r_1-1})\cap(X, Y, \overline{TX_{d-k}}, X_{d-k}, \overline{TX_{d-k-1}}, X_{d-k-1}, ..., \overline{TX_{d-r+1}}, X_{d-r+1})\cap\overline{TX_{d-r_2}} =$$
$$= \sigma^{-1}_{d-r_1}(\overline{TX_{d-r_2}} \cap (X_{d-r_1} \setminus X_{d-r_1-1}))$$

and, in this space, we have

$$\sigma_{d-r_2} \circ \sigma_{d-r_1} = \sigma_{d-r_2}.$$

The following figure shows how the open neighbourhoods TX_{d-r} intersect each other.

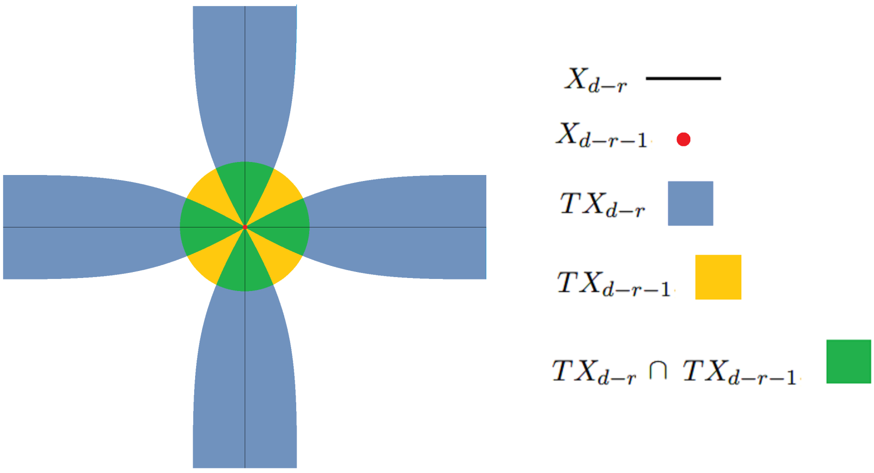

Figure 4.2

Notation 4.2.7. Along this chapter, we will use the superindex ∂ to denote the fibrations of boundaries of suitable tubular naeighborhoods.

Definition 4.2.8. We say that a conical structure verifies the *r-th triviality property* (T_r) if the locally trivial fibration σ^{∂}_{d-r} is trivial, that is, the following two properties hold for every connected component S_{d-r} of $X_{d-r} \setminus X_{d-r-1}$.

(1) There exist an isomorphism

$$(\sigma^{\partial}_{d-r})^{-1}(S_{d-r}) \cong L \times S_{d-r},$$

where L denotes the $2(r-k+1)$-uple of the links of S_{d-r} in

$$(X, Y, \overline{TX_{d-k}}, X_{d-k}, \overline{TX_{d-k-1}}, X_{d-k-1}, ..., \overline{TX_{d-r+1}}, X_{d-r+1}).$$

(2) Under the identification given by property (1), σ^{∂}_{d-r} restricted to $L \times S_{d-r}$ is the canonical projection over S_{d-r}.

Let (X, Y) be a pair of spaces with a conical structure as in Definition 4.2.4 which verifies the *r-th triviality property* (T_r) for any r. Let $k \leq r_1 < r_2 \leq d$ verifying that there exist connected components S_{d-r_1} and S_{d-r_2} of $X_{d-r_1} \setminus X_{d-r_1-1}$ and $X_{d-r_2} \setminus X_{d-r_2-1}$, respectively, such that $(\sigma^{\partial}_{d-r_2})^{-1}(S_{d-r_2}) \cap S_{d-r_1} \neq \emptyset$.

By definition, $(\sigma^{\partial}_{d-r_2})^{-1}(S_{d-r_2}) \cap S_{d-r_1}$, $(\sigma^{\partial}_{d-r_2})^{-1}(S_{d-r_2}) \cap \sigma^{-1}_{d-r_1}(S_{d-r_1})$ and $\sigma^{-1}_{d-r_1}((\sigma^{\partial}_{d-r_2})^{-1}(S_{d-r_2}) \cap S_{d-r_1})$ are $2(k-r+1)$-uplas. To simplify the notation, along the following reasoning we denote by $(\sigma^{\partial}_{d-r_2})^{-1}(S_{d-r_2}) \cap S_{d-r_1}$, $(\sigma^{\partial}_{d-r_2})^{-1}(S_{d-r_2}) \cap \sigma^{-1}_{d-r_1}(S_{d-r_1})$ and $\sigma^{-1}_{d-r_1}((\sigma^{\partial}_{d-r_2})^{-1}(S_{d-r_2}) \cap S_{d-r_1})$ the firsts spaces of these $2(k-r+1)$-uplas.

By the property (T_{r_2}), we know that there are isomorphisms

$$L^{S_{d-r_1}}_{d-r_2} \times S_{d-r_2} \cong (\sigma^{\partial}_{d-r_2})^{-1}(S_{d-r_2}) \cap S_{d-r_1}$$

and

$$L^{\sigma^{-1}_{d-r_1}(S_{d-r_1})}_{d-r_2} \times S_{d-r_2} \cong (\sigma^{\partial}_{d-r_2})^{-1}(S_{d-r_2}) \cap \sigma^{-1}_{d-r_1}(S_{d-r_1})$$

where $L^{S_{d-r_1}}_{d-r_2}$ denotes the link of S_{d-r_2} in S_{d-r_1} and $L^{\sigma^{-1}_{d-r_1}(S_{d-r_1})}_{d-r_2}$ denotes the link of S_{d-r_2} in $\sigma^{-1}_{d-r_1}(S_{d-r_1})$. Moreover, by the property (4) of Definition 4.2.4, we have an isomorphism

$$\sigma^{-1}_{d-r_1}(S_{d-r_1}) \cap (\sigma^{\partial}_{d-r_2})^{-1}(S_{d-r_2}) = \sigma^{-1}_{d-r_1}((\sigma^{\partial}_{d-r_2})^{-1}(S_{d-r_2}) \cap S_{d-r_1})$$

and, using the property (T_{r_1}), we obtain

$$\sigma^{-1}_{d-r_1}((\sigma^{\partial}_{d-r_2})^{-1}(S_{d-r_2}) \cap S_{d-r_1}) \cong c(L^X_{d-r_1}) \times ((\sigma^{\partial}_{d-r_2})^{-1}(S_{d-r_2}) \cap S_{d-r_1})$$

where $L^X_{d-r_1}$ is the link of S_{d-r_1} in X.

Combining the previous isomorphisms, we have

$$L^{\sigma^{-1}_{d-r_1}(S_{d-r_1})}_{d-r_2} \times S_{d-r_2} \cong c(L^X_{d-r_1}) \times ((\sigma^{\partial}_{d-r_2})^{-1}(S_{d-r_2}) \cap S_{d-r_1} \cong c(L^X_{d-r_1}) \times L^{S_{d-r_1}}_{d-r_2} \times S_{d-r_2}$$

Let us denote by γ the isomorphism $c(L^X_{d-r_1}) \times L^{S_{d-r_1}}_{d-r_2} \times S_{d-r_2} \to L^{\sigma^{-1}_{d-r_1}(S_{d-r_1})}_{d-r_2} \times S_{d-r_2}$. Since under the equivalences given by the trivializations the morphisms σ_{d-r_1} and $\sigma^{\partial}_{d-r_2}$ are the canonical projections, using the property (4) of Definition 4.2.4, the diagram

$$\begin{tikzcd}
c(L^X_{d-r_1}) \times L^{S_{d-r_1}}_{d-r_2} \times S_{d-r_2} \arrow[r] \arrow[d, "\gamma"'] & L^{S_{d-r_1}}_{d-r_2} \times S_{d-r_2} \arrow[d] \\
L^{\sigma^{-1}_{d-r_1}(S_{d-r_1})}_{d-r_2} \times S_{d-r_2} \arrow[r] & S_{d-r_2}
\end{tikzcd}$$

where all the morphisms up to γ are the canonical projections, is commutative.

So γ verifies the following condition:

$$\begin{aligned}
\gamma : c(L^X_{d-r_1}) \times L^{S_{d-r_1}}_{d-r_2} \times S_{d-r_2} &\longrightarrow L^{\sigma^{-1}_{d-r_1}(S_{d-r_1})}_{d-r_2} \times S_{d-r_2} \\
(x, y, z) &\longrightarrow (\gamma_1(x, y, z), z)
\end{aligned}$$

We say that the conical structure is *trivial with respect to the connected components* S_{d-r_1} *and* S_{d-r_2} if γ_1 does not depend on z, that is, if there exist an isomorphism $\beta : c(L^X_{d-r_1}) \times L^{S_{d-r_1}}_{d-r_2} \to L^{\sigma^{-1}_{d-r_1}(S_{d-r_1})}_{d-r_2}$ such that $\gamma = (\beta, Id_{S_{d-r_2}})$.

Definition 4.2.9. We say that the conical structure is *trivial* if it verifies the *r-th triviality property* (T_r) for any r and, for every $k \leq r_1 < r_2 \leq d$ and any connected components S_{d-r_1} and S_{d-r_2} of $X_{d-r_1} \setminus X_{d-r_1-1}$ and $X_{d-r_2} \setminus X_{d-r_2-1}$, respectively, either $(\sigma^{\partial}_{d-r_2})^{-1}(S_{d-r_2}) \cap S_{d-r_1} = \emptyset$ or the conical structure is trivial with respect to S_{d-r_1} and S_{d-r_2}.

Remark 4.2.10. For a great variety of topological pseudomanifolds

$$X = X_d \supset X_{d-2} \supset ... \supset X_0 \supset X_{-1} = \emptyset,$$

the pair (X, X_{d-2}) has a conical structure with respect to the stratification

$$X_{d-2} \supset ... \supset X_0 \supset X_{-1} = \emptyset.$$

Whitney stratifications, for instance, verify this property (see [GWPL]). We consider every topological pseudomanifold which appear in sections 4.2, 4.3 and 4.4 holds this. Moreover, we fix such a conical structure and denote the relevant neighbourhoods TX_{d-r} for r varying.

From Section 4.5, it is not necessary to do this assumption

4.2.2 An inductive construction of intersection spaces

Given a topological pseudomanifold we define an inductive procedure on the depth of the strata. The procedure depends on choices made at each inductive step, and may be obstructed for a given set of choices or carried until the deepest stratum. If for a given set of choices can be carried until the end, it produces a pair of spaces which generalizes Banagl intersection spaces.

For our construction we need the notion of fibrewise homology truncation of fibrations of pairs of locally finite CW-complexes:

Definition 4.2.11. Let $\sigma : (X, Y) \to B$ be a locally trivial fibration. We say that σ admits a *fibrewise rational q-homology truncation* if there exits a morphism of pairs of spaces

$$\phi : (X_{\leq q}, Y_{\leq q}) \to (X, Y)$$

such that $\sigma \circ \phi$ is a locally trivial fibration and, for any $b \in B$,

(1) the homomorphism in homology of fibres

$$H_i((X_{\leq q})_b, (Y_{\leq q})_b; \mathbb{Q}) \to H_i(X_b, Y_b; \mathbb{Q})$$

is an isomorphism if $i \leq q$.

(2) the homology group $H_i((X_{\leq q})_b, (Y_{\leq q})_b; \mathbb{Q})$ vanishes if $i > q$.

Notation 4.2.12. Given a pair of spaces (X, Y) we denote by $(X, Y)_{\leq q}$ the pair $(X_{\leq q}, Y_{\leq q})$ appearing in the definition above.

The following figure shows a fibrewise rational 1-homology truncation of the fibration σ in Figure 4.1.

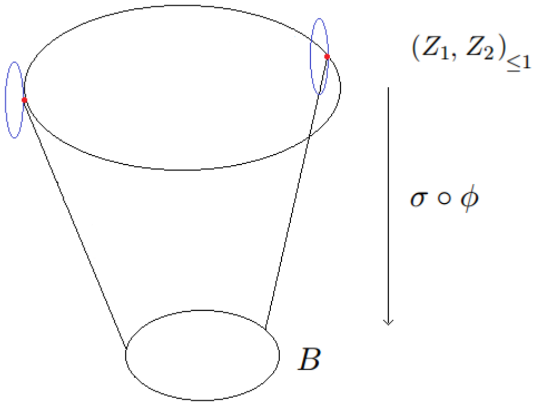

Figure 4.3

Definition 4.2.13. Given a pair of spaces (X, Y) with a conical structure as in Definition 4.2.4, a fibrewise rational q-homology truncation of σ^{∂}_{d-r}

$$\begin{tikzcd}
(\partial\overline{TX_{d-r}} \setminus X_{d-r-1} \cap (X,Y))_{\leq q} \arrow[d, "\phi^{\partial}_{d-r}"] \arrow[dr, "(\sigma^{\partial}_{d-r})_{\leq q}"] & \\
\partial\overline{TX_{d-r}} \setminus X_{d-r-1} \cap (X,Y) \arrow[r, "\sigma^{\partial}_{d-r}"] & X_{d-r} \setminus X_{d-r-1}
\end{tikzcd}$$

is *compatible with the conical structure* if, for every $r' > r$,

$$\sigma_{d-r'} \circ (\sigma^{\partial}_{d-r})_{\leq q} : (\sigma^{\partial}_{d-r})^{-1}_{\leq q}(\overline{TX_{d-r'}} \cap (X_{d-r} \setminus X_{d-r-1})) \to X_{d-r'} \setminus X_{d-r'-1}$$

is the cone of

$$\sigma^{\partial}_{d-r'} \circ (\sigma^{\partial}_{d-r})_{\leq q} : (\sigma^{\partial}_{d-r})^{-1}_{\leq q}(\partial\overline{TX_{d-r'}} \cap (X_{d-r} \setminus X_{d-r-1})) \to X_{d-r'} \setminus X_{d-r'-1}$$

Proposition 4.2.14. *Given a pair of spaces (X, Y) with a conical structure as in Definition 4.2.4, if there exist a fibrewise rational q-homology truncation of σ^{∂}_{d-r}, then there exist a fibrewise rational q-homology truncation of σ^{∂}_{d-r} compatible with the conical structure.*

Proof. Let us consider a fibrewise rational q-homology truncation of σ^{∂}_{d-r}, $(\sigma^{\partial}_{d-r})_{\leq q}$.

Then, the restriction of $(\sigma^{\partial}_{d-r})_{\leq q}$ to

$$(\sigma^{\partial}_{d-r})^{-1}_{\leq q}((X_{d-r} \setminus X_{d-r-1}) \setminus \bigcup_{r'>r} TX_{d-r'})$$

is a fibrewise rational q-homology truncation of the restriction of σ^{∂}_{d-r} to

$$(\sigma^{\partial}_{d-r})^{-1}((X_{d-r} \setminus X_{d-r-1}) \setminus \bigcup_{r'>r} TX_{d-r'})$$

Using the property (2) of Definition 4.2.4, we can extend the previous restriction to a fibrewise rational q-homology truncation over

$$(X_{d-r} \setminus X_{d-r-1}) \setminus \bigcup_{r''>r'} TX_{d-r''}$$

for any $r' > r$ inductively. Moreover, using the property (3) of Definition 4.2.4, we can check that these extensions are compatible with the conical structure. So, when $r' = d$, we obtain a fibrewise rational q-homology truncation of σ^{∂}_{d-r} compatible with the conical structure. □

Definition 4.2.15. Given a pair of spaces (X, Y) with a conical structure verifying the triviality property (T_r) for any r (see Definition 4.2.8), a fibrewise rational q-homology truncation of σ^{∂}_{d-r}

$$\begin{tikzcd}
(\partial\overline{TX_{d-r}} \setminus X_{d-r-1} \cap (X,Y))_{\leq q} \arrow[d, "\phi^{\partial}_{d-r}"] \arrow[dr, "(\sigma^{\partial}_{d-r})_{\leq q}"] & \\
\partial\overline{TX_{d-r}} \setminus X_{d-r-1} \cap (X,Y) \arrow[r, "\sigma^{\partial}_{d-r}"'] & X_{d-r} \setminus X_{d-r-1}
\end{tikzcd}$$

is *compatible with the trivialization* if the following conditions hold.

(1) Given a connected component S_{d-r} of $X_{d-r} \setminus X_{d-r-1}$, if $L_{d-r} = (L^X_{d-r}, L^Y_{d-r})$ denotes the link of S_{d-r} in (X, Y) and

$$(\sigma^{\partial}_{d-r})^{-1}(S_{d-r}) \cong L_{d-r} \times S_{d-r}$$

is the isomorphism induced by property (T_r), then there exist a pair of spaces $(L_{d-r})_{\leq q} := ((L^X_{d-r})_{\leq q}, (L^Y_{d-r})_{\leq q})$ such that the homology group $H_i((L^X_{d-r})_{\leq q}, (L^Y_{d-r})_{\leq q}; \mathbb{Q})$ vanishes if $i > q$, we have an isomorphism

$$(\sigma^{\partial}_{d-r})^{-1}_{\leq q}(S_{d-r}) \cong (L_{d-r})_{\leq q} \times S_{d-r}$$

and, under these identifications, $((\sigma^{\partial}_{d-r})_{\leq q})_{|(L_{d-r})_{\leq q}\times S_{d-r}}$ is the canonical projection and $(\phi^{\partial}_{d-r})_{|(L_{d-r})_{\leq q}\times S_{d-r}} = (\phi_1, Id_{S_{d-r}})$ where $\phi_1 : (L_{d-r})_{\leq q} \to L_{d-r}$ is a morphism such that

$$H_i(\phi_1) : H_i((L^X_{d-r})_{\leq q}, (L^Y_{d-r})_{\leq q}; \mathbb{Q}) \to H_i(L^X_{d-r}, L^Y_{d-r}; \mathbb{Q})$$

is an isomorphism if $i \leq q$.

(2) Given $r' > r$ and a connected component $S_{d-r'}$ of $X_{d-r'} \setminus X_{d-r'-1}$ such that

$$(\sigma^{\partial}_{d-r'})^{-1}(S_{d-r}) \cap S_{d-r} \neq \emptyset,$$

let $L^{S_{d-r}}_{d-r'}$ and $L^{\sigma^{-1}_{d-r}(S_{d-r})}_{d-r'}$ denote the link of $S_{d-r'}$ in S_{d-r} and $\sigma^{-1}_{d-r}(S_{d-r})$ respectively. Moreover, let

$$\gamma : c(L^X_{d-r}) \times L^{S_{d-r}}_{d-r'} \times S_{d-r'} \cong L^{\sigma^{-1}_{d-r}(S_{d-r})}_{d-r'} \times S_{d-r'}$$

be the isomorphism defined in the previous section. Then, the image of the composition

$$\begin{tikzcd} c((L^X_{d-r})_{\leq q}) \times L^{S_{d-r}}_{d-r'} \times S_{d-r'} \arrow[r, "(c(\phi_1),Id)"] & c(L^X_{d-r}) \times L^{S_{d-r}}_{d-r'} \times S_{d-r'} \arrow[d, "\gamma"] \\ & L^{\sigma^{-1}_{d-r}(S_{d-r})}_{d-r'} \times S_{d-r'} \end{tikzcd}$$

is equal to $A \times S_{d-r'}$ for some subset $A \subset L^{\sigma^{-1}_{d-r}(S_{d-r})}_{d-r'}$.

Remark 4.2.16. If the conical structure is trivial (see Definition 4.2.9), the condition (1) of the previous definition implies the condition (2).

Remark 4.2.17. If a fibrewise rational q-homology truncation of σ^{∂}_{d-r} is compatible with the trivialization, then it is also compatible with the conical structure.

The initial step of the induction.

Let X be a topological pseudomanifold such that the pair (X, X_{d-2}) has a conical structure with respect to the stratification, we consider the open neighbourhoods TX_{d-r} fixed in Remark 4.2.10.

Let m be the minimum such that $X_{d-m} \setminus X_{d-m-1} \neq \emptyset$. If the fibration

$$\sigma^{\partial}_{d-m} : \partial\overline{TX_{d-m}} \setminus X_{d-m-1} \to X_{d-m} \setminus X_{d-m-1}$$

predicted in Definition 4.2.4 does not admit a fibrewise rational $\bar{q}(m)$-homology truncation, then the intersection space does not exist. Otherwise we choose a fibrewise rational $\bar{q}(m)$-homology truncation compatible with the conical structure (see Proposition 4.2.14)

$$\begin{tikzcd}
(\partial \overline{TX_{d-m}} \setminus X_{d-m-1})_{\leq \bar{q}(m)} \arrow[dd, "\phi^{\partial}_{d-m}"'] \arrow[dr, "(\sigma^{\partial}_{d-m})_{\leq \bar{q}(m)}"] & \\
& X_{d-m} \setminus X_{d-m-1} \\
\partial \overline{TX_{d-m}} \setminus X_{d-m-1} \arrow[ur, "\sigma^{\partial}_{d-m}"] &
\end{tikzcd}$$

We construct a new space X'_m, a homotopy equivalence $\pi_m : X'_m \to X$ with contractible fibres and a subspace $I^{\bar{p}}_m X \hookrightarrow X'_m$ as follows.

Define the map

$$(\sigma_{d-m})_{\leq \bar{q}(m)} : cyl((\sigma^{\partial}_{d-m})_{\leq \bar{q}(m)}) \to X_{d-m} \setminus X_{d-m-1}$$

to be the cone of the fibration $(\sigma^{\partial}_{d-m})_{\leq \bar{q}(m)}$ over $X_{d-m} \setminus X_{d-m-1}$. By property (2) of Definition 4.2.4 there exits a fibre bundle morphism

$$\phi_{d-m} : cyl((\sigma^{\partial}_{d-m})_{\leq \bar{q}(m)}) \to \overline{TX_{d-m}} \setminus X_{d-m-1}$$

over the base $X_{d-m} \setminus X_{d-m-1}$ which preserves the vertex sections. Let

$$\theta_{d-m} : cyl((\sigma^{\partial}_{d-m})_{\leq \bar{q}(m)}) \to X \setminus X_{d-m-1}$$

be the composition of the fibre bundle morphism ϕ_{d-m} with the natural inclusion of the closed subset $\overline{TX_{d-m}} \setminus X_{d-m-1}$ into $X \setminus X_{d-m-1}$. Let $\mathrm{cyl}(\theta_{d-m})$ be the mapping cylinder of θ_{d-m}. It is by definition the union $cyl((\sigma^{\partial}_{d-m})_{\leq \bar{q}(m)}) \times [0,1] \coprod (X \setminus X_{d-m-1})$ under the usual equivalence relation. Denote by

$$s_{d-m} : X_{d-m} \setminus X_{d-m-1} \to cyl((\sigma^{\partial}_{d-m})_{\leq \bar{q}(m)})$$

the vertex section. We define Z_m to be the result of quotienting $\mathrm{cyl}(\theta_{d-m})$ by the equivalence relation which identifies, for any $x \in X_{d-m} \setminus X_{d-m-1}$, the subspace $s_{d-m}(x) \times [0,1]$ to a point.

The following figures show $\mathrm{cyl}(\theta_{d-m})$ and Z_m.

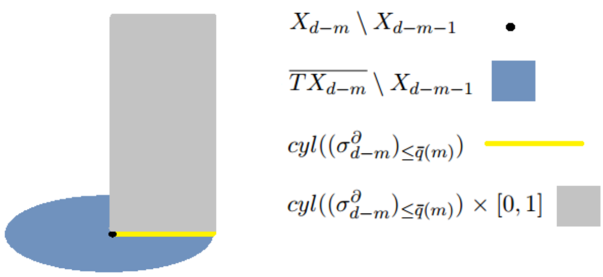

Figure 4.4: $\mathrm{cyl}(\theta_{d-m})$

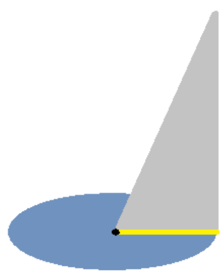

Figure 4.5: Z_m

We have a natural projection map $\pi_m : Z_m \to X \setminus X_{d-m-1}$ which is a homotopy equivalence whose fibres are contractible and has a natural section denoted by α_m. α_m is a closed inclusion of $X \setminus X_{d-m-1}$ into Z_m. In the previous figure, π_m is the projection onto the plane and α_m is the inclusion of the plane.

Define X'_m as the set $Z_m \cup X_{d-m-1}$. The projection map extends to a projection

$$\pi_m : X'_m \to X$$

Consider in X'_m the topology spanned by all the open subsets of Z_m and the collection of subsets of the form $\pi_m^{-1}(U)$ for any open subset U of X.

This projection is also a homotopy equivalence whose fibres are contractible and such that the natural section α_m extends to it giving a closed inclusion of X into X'_m.

Define the step m intersection space $I^{\bar{p}}_m X$ to be the subspace of X'_m given by

$$I^{\bar{p}}_m X := cyl((\sigma^{\partial}_{d-m})_{\leq \bar{q}(m)}) \times \{0\} \cup cyl(\phi^{\partial}_{d-m}) \cup (X \setminus TX_{d-m}),$$

with the restricted topology.

The following figure shows $I^{\bar{p}}_m X$.

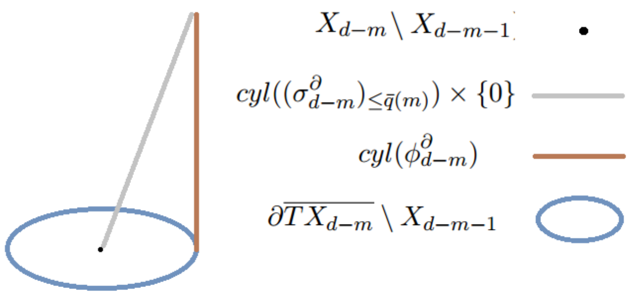

Figure 4.6

Remark 4.2.18. Note that $cyl(\phi^{\partial}_{d-m}) \cong \pi_m^{-1}(\partial\overline{TX_{d-m}} \setminus X_{d-m-1})$.

With the above definitions we have the following chains of inclusions

$$X'_m \supset X \supset X_{d-m} \supset ... \supset X_0,$$

$$X'_m \supset I^{\bar{p}}_m X \supset X_{d-m} \supset ... \supset X_0,$$

where X is embedded in X'_m via the section α_m.

An immediate consequence of our construction is

Lemma 4.2.19. *The pairs (X'_m, X_{d-m}) and $(I^{\bar{p}}_m X, X_{d-m})$ have a conical structure with respect to the stratified subspace $X_{d-m-1} \supset ... \supset X_0$, given by the open neighbourhoods of $X_{d-r} \setminus X_{d-r-1}$ $\pi_m^{-1}(TX_{d-r})$ in $X'_m \setminus X_{d-r-1}$ and $\pi_m^{-1}(TX_{d-r}) \cap I^{\bar{p}}_m X$ in $I^{\bar{p}}_m X \setminus X_{d-r-1}$.*

The inductive step.

At this point we are ready to set up the inductive step of the construction of intersection spaces. The inductive step is different in nature to the initial step in the following sense. The necessary condition to be able to carry the initial step is that a link fibration admits a fibrewise rational q-homology truncation. In the inductive step, this condition is replaced by the fact that a fibration of link pairs admits a fibrewise rational q-homology truncation. The smaller space in the pair is constructed by iterated modifications of $X_{d-2} = X_{d-m}$. Define

$$I^{\bar{p}}_m(X_{d-2}) := X_{d-2}.$$

Remark 4.2.20. Note that $I^{\bar{p}}_m(X_{d-2})$ is different from the step m intersection space of X_{d-2}, $I^{\bar{p}}_m X_{d-2}$, if it exists.

We assume by induction that, for $k \geq m$, we have constructed

(i) a space X'_k and a projection
$$\pi_k : X'_k \to X$$
which is a homotopy equivalence with contractible fibres, together with a section α_k providing a closed inclusion of X into X'_k.

(ii) subspaces $I^{\bar{p}}_k(X_{d-2}) \subset I^{\bar{p}}_k X \subset X'_k$ such that, embedding X into X'_k via α_k, we have the topological pseudomanifold
$$X_0 \subset X_1 \subset ... \subset X_{d-k-1}$$
embedded into $I^{\bar{p}}_k(X_{d-2})$,

(iii) the pairs $(X'_k, I^{\bar{p}}_k X)$, $(I^{\bar{p}}_k X, I^{\bar{p}}_k(X_{d-2}))$ have respective conical structures with respect to the stratified subspace described in the previous point. The open neighbourhoods of $X_{d-r} \setminus X_{d-r-1}$ appearing in these structures are $\pi_k^{-1}(TX_{d-r})$ in $X'_k \setminus X_{d-k-1}$ and $\pi_k^{-1}(TX_{d-r}) \cap I^{\bar{p}}_k X$ in $I^{\bar{p}}_k X \setminus X_{d-k-1}$ respectively.

If $X_{d-k-1} \setminus X_{d-k-2}$ is empty we define $X'_{k+1} := X'_k$, $\pi_{k+1} := \pi'_k$, $\alpha_{k+1} := \alpha_k$, $I^{\bar{p}}_{k+1} X := I^{\bar{p}}_k X$, and $I^{\bar{p}}_{k+1}(X_{d-2}) := I^{\bar{p}}_k(X_{d-2})$. It is clear that the required conditions are satisfied.

If $X_{d-k-1} \setminus X_{d-k-2}$ is not empty, since the pair $(I^{\bar{p}}_k X, I^{\bar{p}}_k(X_{d-2}))$ has a conical structure with respect to the stratified subspace

$$X_0 \subset X_1 \subset ... \subset X_{d-k-1},$$

we have a locally trivial fibration of pairs

$$\sigma^{\partial}_{d-k-1} : (\partial\overline{\pi_k^{-1}(TX_{d-k-1})} \setminus X_{d-k-2}) \cap (I^{\bar{p}}_k X, I^{\bar{p}}_k(X_{d-2})) \to X_{d-k-1} \setminus X_{d-k-2}.$$

If the fibration does not admit a fibrewise rational $\bar{q}(k+1)$-homology truncation, then the intersection space construction cannot be completed with the previous choices.

Otherwise we choose a fibrewise rational $\bar{q}(k+1)$-homology truncation compatible with the conical structure (see Proposition 4.2.14)

$$\begin{tikzcd}
((\overline{\partial\pi_k^{-1}(TX_{d-k-1})} \setminus X_{d-k-2}) \cap (I_k^{\bar{p}}X, I_k^{\bar{p}}(X_{d-2}))_{\leq \bar{q}(k+1)} \arrow[d, "\phi^{\partial}_{d-k-1}"] \arrow[dr, "(\sigma^{\partial}_{d-k-1})_{\leq \bar{q}(k+1)}"] & \\
& X_{d-k-1} \setminus X_{d-k-2} \\
(\overline{\partial\pi_k^{-1}(TX_{d-k-1})} \setminus X_{d-k-2}) \cap (I_k^{\bar{p}}X, I_k^{\bar{p}}(X_{d-2})) \arrow[ur, "\sigma^{\partial}_{d-k-1}"'] &
\end{tikzcd}$$

We construct now a homotopy equivalence $\pi_{k+1} : X'_{k+1} \to X$ with contractible fibres and a pair of subspaces $(I^{\bar{p}}_{k+1}X, I^{\bar{p}}_{k+1}(X_{d-2})) \hookrightarrow X'_{k+1}$ as follows.

Let

$$(\sigma_{d-k-2})_{\leq \bar{q}(k+1)} : cyl((\sigma^{\partial}_{d-k-1})_{\leq \bar{q}(k+1)}) \to X_{d-k-1} \setminus X_{d-k-2}$$

be the cone of the fibration of pairs $(\sigma^{\partial}_{d-k-1})_{\leq \bar{q}(k+1)}$ over $X_{d-k-1} \setminus X_{d-k-2}$. Recall that, according with Definition 4.2.3, $cyl((\sigma^{\partial}_{d-k-1})_{\leq \bar{q}(k+1)})$ is a pair of spaces.

By property (2) of Definition 4.2.4 there exits a morphism of fibre bundles of pairs

$$\phi_{d-k-1} : cyl((\sigma^{\partial}_{d-k-1})_{\leq \bar{q}(k+1)}) \to (\overline{\pi_k^{-1}(TX_{d-k-1})} \setminus X_{d-k-2}) \cap (I_k^{\bar{p}}X, I_k^{\bar{p}}(X_{d-2}))$$

over the base $X_{d-k-1} \setminus X_{d-k-2}$ which preserves the vertex sections.

Let

$$\theta_{d-k-1} : cyl((\sigma^{\partial}_{d-k-1})_{\leq \bar{q}(k+1)}) \to X'_k \setminus X_{d-k-2}$$

be the composition of the fibre bundle morphism ϕ_{d-k-1} with the natural inclusion

$$(\overline{\pi_k^{-1}(TX_{d-k-1})} \setminus X_{d-k-2}) \cap (I_k^{\bar{p}}X, I_k^{\bar{p}}(X_{d-2})) \hookrightarrow X'_k \setminus X_{d-k-2}.$$

Let $\mathrm{cyl}(\theta_{d-k-1})$ be the mapping cylinder of θ_{d-k-1}. It is by definition the union of the pair $cyl((\sigma^{\partial}_{d-k-1})_{\leq \bar{q}(k+1)}) \times [0,1]$ with the pair $(X'_k \setminus X_{d-k-2}, \mathrm{Im}(\phi_{d-k-1})_2)$ with the usual equivalence relation.

Denote by

$$s_{d-k-1} : X_{d-k-1} \setminus X_{d-k-2} \to cyl((\sigma^{\partial}_{d-k-1})_{\leq \bar{q}(k+1)})$$

the vertex section. We define Z_{k+1} to be the pair of spaces which results of quotienting $\mathrm{cyl}(\theta_{d-k-1})$ by the equivalence relation which identifies, for any $x \in X_{d-k-1} \setminus X_{d-k-2}$, the subspace $s_{d-k-1}(x) \times [0,1]$ to a point.

We denote the spaces forming the pair Z_{k+1} by $Z_{k+1} = (Z^1_{k+1}, Z^2_{k+1})$. We have a natural projection map $\rho_{k+1} : Z^1_{k+1} \to X'_k \setminus X_{d-k-2}$ which is a homotopy equivalence whose fibres are contractible, and has a natural section denoted by β_{k+1}. The composition

$$\pi_{k+1} := \pi_k|_{X'_k \setminus X_{d-k-2}} \circ \rho_{k+1} : Z^1_{k+1} \to X \setminus X_{d-k-2}$$

is a homotopy equivalence with contractible fibres, and has a section $\alpha_{k+1} := \beta_{k+1} \circ \alpha_k|_{X \setminus X_{d-k-2}}$ providing a closed inclusion of $X \setminus X_{d-k-2}$ into Z^1_{k+1}.

Define X'_{k+1} as the set $Z^1_{k+1} \cup X_{d-k-2}$. The projection maps ρ_{k+1} and π_{k+1} extends to projections

$$\rho_{k+1} : X'_{k+1} \to X'_k$$
$$\pi_{k+1} : X'_{k+1} \to X.$$

Consider the topology in X'_{k+1} spanned by the all the open subsets of Z_{k+1} and the collection of subsets of the form $\pi^{-1}_{k+1}(U)$ for any open subset U of X. With this topology the projections are also homotopy equivalences whose fibres are contractible, and such that the natural sections β_{k+1} and α_{k+1} extend to them as closed inclusions.

Define the step $k+1$ intersection space pair to be the pair of subspaces of X'_{k+1} given by

$$(I^{\bar{p}}_{k+1}X, I^{\bar{p}}_{k+1}(X_{d-2})) := cyl((\sigma^{\partial}_{d-k-1})_{\leq \bar{q}(k)}) \times \{0\} \cup$$
$$\cup cyl(\phi^{\partial}_{d-k-1}) \cup (I^{\bar{p}}_k X, I^{\bar{p}}_k(X_{d-2})) \setminus \pi_k^{-1}(TX_{d-k-1}),$$

with the restricted topology.

Remark 4.2.21. Note that $cyl(\phi^{\partial}_{d-k-1}) \cong \pi^{-1}_{k+1}(\partial\overline{TX_{d-k-1}} \setminus X_{d-k-2})$.

With the definitions above it is easy to check that conditions (i)-(iii) are satisfied replacing k by $k+1$ and the induction step is complete.

Definition 4.2.22. Given a topological pseudomanifold $X_d \supset ... \supset X_0$ such that the pair (X_d, X_{d-2}) has a conical structure with respect to the stratification (see Definition 4.2.15 and Remark 4.2.10), we say that it has an intersection space pair if there exist successive choices of suitable fibrewise homology truncations so that the construction above can be carried up to $k = d$. In that case the pair

$$(I^{\bar{p}}X, I^{\bar{p}}(X_{d-2})) = (I^{\bar{p}}_d X, I^{\bar{p}}_d(X_{d-2}))$$

is called *an intersection space pair associated with the stratification.*

Definition 4.2.23. We denote $X' := X'_d$. The *homotopy model* of X is the homotopy equivalence π_d which we denote $\pi : X' \to X$. The section α_d is denoted by $\alpha : X \to X'$ and provides a closed inclusion of X into X'.

Remark 4.2.24. If the intersection space pair exists it does not have to be unique up to homotopy. The different choices of fibrewise homology truncations may yield different choices of intersection spaces. The construction of intersection spaces follow the scheme of obstruction theory in algebraic topology: previous choices of fibrewise homology truncation may affect the possibility of finishing the construction in the subsequent steps.

Proposition 4.2.25. *If X is a topological pseudomanifold with a trivial conical structure (see Definition 4.2.9) and we choose fibrewise homology truncations which are compatible with the trivialization along our construction (see Definition 4.2.15), then the triviality of the conical structure is preserved in every step.*

Given any pair of spaces obtained at some step with our construction such that it has a trivial conical structure, we can construct a fibrewise rational q-homology truncation of the fibration σ^{∂}_{d-r} in one fibre for any r and any q with the methods in [Ban10, Chapter 1] and, afterwards, we can extend this truncation using the trivialization. The resultant fibrewise homology truncation of σ^{∂}_{d-r} is compatible with the trivialization. So, we get the following result.

Corollary 4.2.26. *Let X be a topological pseudomanifold with a trivial conical structure. Then, X has an intersection space pair for every perversity.*

Example 4.2.27. Toric varieties have conical structures which are trivial. So, every toric variety has an intersection space pair for every perversity.

4.3 A sequence of Intersection Space pairs

Our aim is to associate with any choice of intersection space pair, a constructible complex on the original topological pseudomanifold X, whose hypercohomology coincides with the hypercohomology of the intersection space $I^{\bar{p}}X$. In order to do so, we define an increasing sequence of modified intersection space pairs, all of them included in the homotopy model X'. We provide precise definitions of the sequence, but leave many of the straightforward checking to the reader.

4.3.1 Systems of neighborhoods.

Given a topological pseudomanifold

$$X = X_d \supset X_{d-2} \supset ... \supset X_0 \supset X_{-1} = \emptyset,$$

such that the pair (X, X_{d-2}) has a conical structure with respect to the stratification

$$X_{d-2} \supset ... \supset X_0 \supset X_{-1} = \emptyset$$

(see Remark 4.2.10), we denote the relevant neighbourhoods by TX_{d-r} for r varying.

Property (2) of Definition 4.2.4 states that the fibration σ_{d-r} is the cone of the fibration σ^{∂}_{d-r} over the base $X_{d-r} \backslash X_{d-r-1}$. This means precisely that $\overline{TXd-r} \backslash X_{d-r-1}$ is equal to the product

$$\partial\overline{TX_{d-r}} \setminus X_{d-r-1} \times [0,1],$$

modulo the equivalence relation which identifies $(x, 1)$ and $(y, 1)$ if $\sigma^{\partial}_{d-r}(x)$ equals $\sigma^{\partial}_{d-r}(y)$.

For any $r \in 2, ..., d$ and any $n \in \mathbb{N}$ we define the open neighborhood $T^n X_{d-r}$ to be the quotient of

$$\partial\overline{TX_{d-r}} \setminus X_{d-r-1} \times (1 - 1/(n+1), 1]$$

under the same equivalence relation.

The open subsets $T^n X_{d-r}$ for n varying, form a system of tubular neighborhoods of $X_{d-r} \setminus X_{d-r-1}$ in $X \setminus X_{d-r-1}$, whose intersection is the stratum $X_{d-r} \setminus X_{d-r-1}$. Moreover, for any fixed n the collection of neighborhoods $T^n X_{d-r}$, for r varying, give a conical structure to (X, X_{d-2}) with respect to the topological pseudomanifold $X_{d-2} \supset ... \supset X_0$.

4.3.2 The sequence of intersection space pairs.

Suppose that there exists successive choices of suitable fibrewise homology truncations so that the construction of intersection space pairs described in the previous section can be carried up to $k = d$. Fix such a choice.

Fix $n \in \mathbb{N}$. Following the inductive construction of the previous section we produce a sequence of pairs

$$(I_k^{\bar{p},n} X, I_k^{\bar{p},n}(X_{d-2}))$$

for $k = 2, ..., r$ as follows.

Let m be the minimum such that $X_{d-m} \setminus X_{d-m-1} \neq \emptyset$. Define

$$K_m^n(X) := \pi_m^{-1}(\overline{TX_{d-m}} \setminus T^n X_{d-m}) \subset X'_m,$$

$$I_m^{\bar{p},n}(X) := I_m^{\bar{p}}(X) \cup K_m^n(X),$$

$$C_m^n(X_{d-2}) := \emptyset,$$

$$I_m^{\bar{p},n}(X_{d-2}) := I_m^{\bar{p}}(X_{d-2}) = X_{d-2}.$$

Remark 4.3.1. Note that $\pi_m^{-1}((\overline{TX_{d-m}} \setminus T^n X_{d-m}) \setminus X_{d-m-1}) \cong cyl(\phi_{d-m}^{\partial}) \times [0, 1 - 1/(n+1)]$ (see Remark 4.2.18).

The following figure shows the previous modification in Figure 4.6. $K_m^n(X)$ is the union of blue set and brown set.

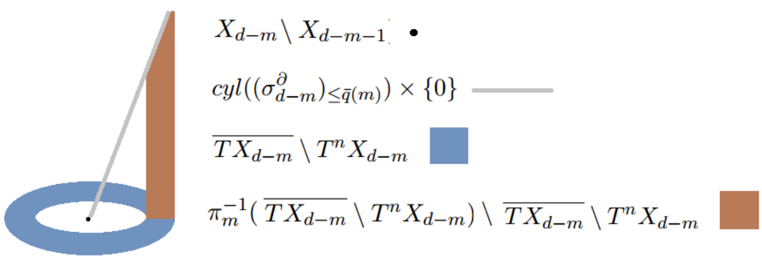

Figure 4.7

The following figure shows $\overline{TX_{d-m}} \setminus T^n X_{d-m}$ with more dimension than in the previous figure. $K_m^n(X)$ is the preimage of this set by the morphism π_m.

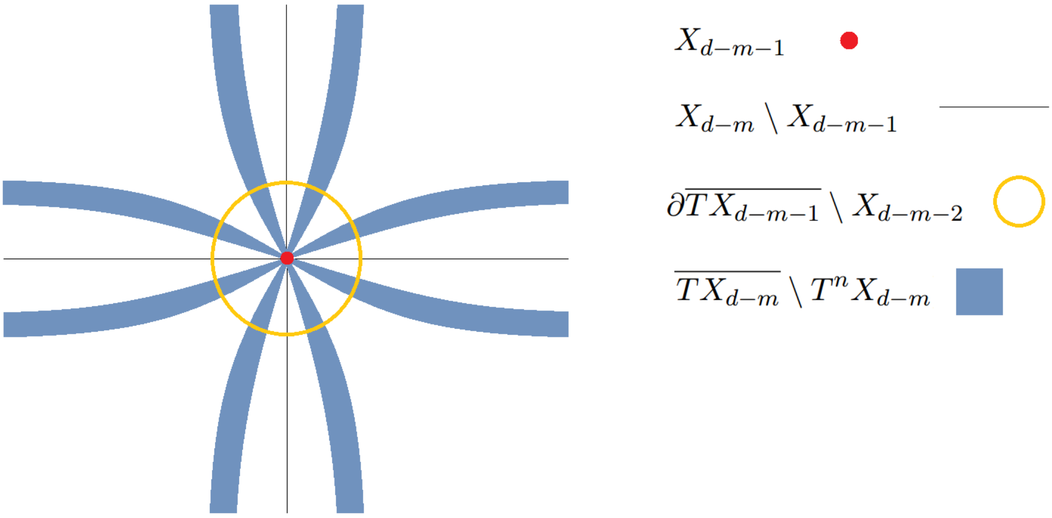

Figure 4.8

Assume that $K_k^n(X), C_k^n(X_{d-2}), I_k^{\bar{p},n}(X)$ and $I_k^{\bar{p},n}(X_{d-2})$ have been defined. Define

$$K_{k+1}^n(X) := \rho_{k+1}^{-1}\big(((I_k^{\bar{p}}X \cap \pi_k^{-1}(\overline{TX_{d-(k+1)}})) \setminus \pi_k^{-1}(T^n X_{d-(k+1)}))\cup$$

$$\cup(K_k^n(X) \setminus \pi_k^{-1}(T^n X_{d-(k+1)}))\big),$$

$$I_{k+1}^{\bar{p},n}X := I_{k+1}^{\bar{p}}X \cup K_{k+1}^n(X),$$

$$C_{k+1}^n(X_{d-2}) := \rho_{k+1}^{-1}\big(((I_k^{\bar{p}}(X_{d-2}) \cap \pi_k^{-1}(\overline{TX_{d-(k+1)}})) \setminus \pi_k^{-1}(T^n X_{d-(k+1)}))\cup$$

$$\cup(C_k^n(X_{d-2}) \setminus \pi_k^{-1}(T^n X_{d-(k+1)}))\big),$$

$$I_{k+1}^{\bar{p},n}(X_{d-2}) := I_{k+1}^{\bar{p}}(X_{d-2}) \cup C_{k+1}^n(X_{d-2}).$$

The following figure shows $(\overline{TX_{d-m}} \setminus T^n X_{d-m}) \setminus T^n X_{d-m-1}$ in blue and green and $(\overline{TX_{d-m-1}} \setminus T^n X_{d-m-1}) \setminus (\overline{TX_{d-m}} \setminus T^n X_{d-m})$ in yellow. $K_{m+1}^n(X)$ is the union of

- the preimage of the blue and green set by π_{m+1}
- the preimage of the yellow set by π_{m+1} intersected with the preimage of $I_m^{\bar{p}}X$ by ρ_{m+1}

$C_{m+1}^n(X_{d-2})$ is the union of

- the preimage of the blue and green set by π_{m+1}
- the preimage of the yellow set by π_{m+1} intersected with the preimage of $I_m^{\bar{p}}(X_{d-2})$ by ρ_{m+1}

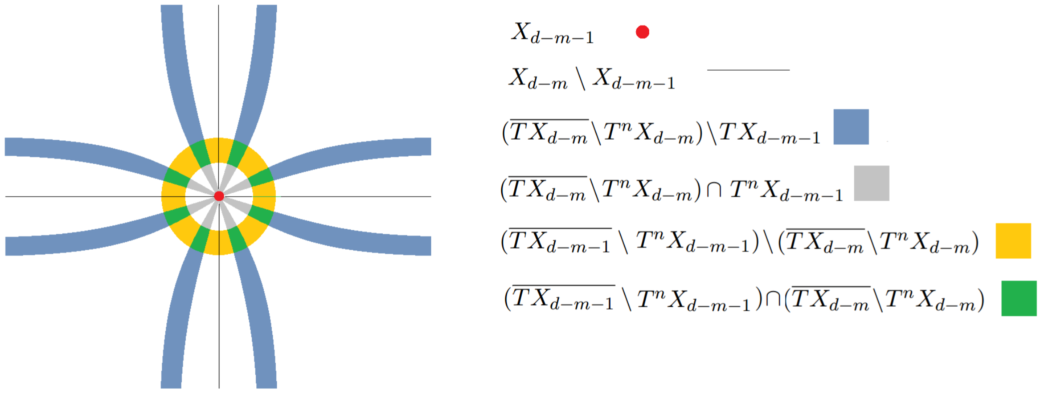

Figure 4.9

Iterate the construction until $k = d$ and define

$$(I^{\bar{p},n}X, I^{\bar{p},n}(X_{d-2})) := (I_d^{\bar{p},n}X, I_d^{\bar{p},n}(X_{d-2})),$$

which is a pair of subsets of X'.

Since the closed subsets $K_k^n(X), C_k^n(X_{d-2})$ are increasingly larger when n increases we have constructed a sequence of pairs of closed subsets

$$(I^{\bar{p}}X, I^{\bar{p}}(X_{d-2})) \subset ... \subset (I^{\bar{p},n}X, I^{\bar{p},n}(X_{d-2})) \subset (I^{\bar{p},n+1}X, I^{\bar{p},n+1}(X_{d-2})) \subset ...$$

An easy inspection on the construction shows:

Proposition 4.3.2. *The previous construction has the following properties.*

(1) *The inclusions $I^{\bar{p},n}X \subset I^{\bar{p},n+1}X$ and $I^{\bar{p},n}(X_{d-2}) \subset I^{\bar{p},n+1}(X_{d-2})$ are strong deformation retracts for any $n \in \mathbb{N}$. If we denote the inclusions by $\nu^n_{X_{d-2}}$: $I^{\bar{p},n}(X_{d-2}) \to I^{\bar{p},n}X$, $i^n_X : I^{\bar{p},n}X \to I^{\bar{p},n+1}X$ and $i^n_{X_{d-2}} : I^{\bar{p},n}(X_{d-2}) \to I^{\bar{p},n+1}(X_{d-2})$ and the retractions by $r^n_X : I^{\bar{p},n+1}X \to I^{\bar{p},n}X$ and $r^n_{X_{d-2}} : I^{\bar{p},n+1}(X_{d-2}) \to I^{\bar{p},n}(X_{d-2})$, we have commutative diagrams*

$$\begin{tikzcd}
I^{\bar{p},n}(X_{d-2}) \arrow[r, "\nu^n"] \arrow[d, shift left, "i^n_{X_{d-2}}"] & I^{\bar{p},n}X \arrow[d, shift right, "i^n_X"'] \\
I^{\bar{p},n+1}(X_{d-2}) \arrow[r, "\nu^{n+1}"] \arrow[u, shift left, "r^n_{X_{d-2}}"] & I^{\bar{p},n+1}X \arrow[u, shift right, "r^n_X"']
\end{tikzcd} \tag{4.1}$$

(2) *We have the equality $I^{\bar{p},n}(X_{d-2}) = I^{\bar{p},n+1}(X_{d-2}) \cap I^{\bar{p},n}(X)$.*

(3) *For any $x \in X \setminus X_{d-2}$, there exist a small neighbourhood U_x of x in X and a natural number n_0 such that, for every $n > n_0$, $\pi^{-1}(U_x)$ is contained in $I^{\bar{p},n}X$ and $\pi^{-1}(U_x) \cap I^{\bar{p},n}(X_{d-2}) = \emptyset$.*

(4) *For any $x \in X_{d-r} \setminus X_{d-r-1}$, there exists a small neighbourhood U_x of x in X and a natural number n_0 such that, for any $n > n_0$, the diagram (4.1) restricts to the diagram*

$$\begin{tikzcd} I^{\bar{p},n}(X_{d-2}) \cap \pi^{-1}(U_x) \arrow[r, "\nu^n"] \arrow[d, shift left, "i^n_{X_{d-2}}"] & I^{\bar{p},n} X \cap \pi^{-1}(U_x) \arrow[d, shift right, "i^n_X"'] \\ I^{\bar{p},n+1}(X_{d-2}) \cap \pi^{-1}(U_x) \arrow[r, "\nu^{n+1}"] \arrow[u, shift left, "r^n_{X_{d-2}}"] & I^{\bar{p},+1n} X \cap \pi^{-1}(U_x) \arrow[u, shift right, "r^n_X"'] \end{tikzcd} \tag{4.2}$$

and we have the equalities $r^n_{X_{d-2}}(I^{\bar{p},n+1}(X_{d-2}) \cap \pi^{-1}(U_x)) = I^{\bar{p},n}(X_{d-2}) \cap \pi^{-1}(U_x)$ and $r^n_X(I^{\bar{p},n+1}(X) \cap \pi^{-1}(U_x)) = I^{\bar{p},n}(X) \cap \pi^{-1}(U_x)$.
Then, the inclusions

$$\pi^{-1}(U_x) \cap I^{\bar{p},n_1} X \hookrightarrow \pi^{-1}(U_x) \cap I^{\bar{p},n_2} X,$$
$$\pi^{-1}(U_x) \cap I^{\bar{p},n_1}(X_{n-2}) \hookrightarrow \pi^{-1}(U_x) \cap I^{\bar{p},n_2}(X_{n-2})$$

are strong deformation retracts.

Sketch of the proof. For any r, we have an equality

$$(\overline{TX_{d-r}} \setminus T^n X_{d-r}) \setminus X_{d-r-1} = \partial \overline{TX_{d-r}} \setminus X_{d-r-1} \times [0, 1 - 1/(n+1)].$$

So, there are canonical retractions $\overline{TX_{d-r}} \setminus T^{n+1} X_{d-r} \to \overline{TX_{d-r}} \setminus T^n X_{d-r}$. These retractions induce retractions $K^{n+1}_r(X) \to K^n_r(X)$ and $C^{n+1}_r(X_{d-2}) \to C^n_r(X_{d-2})$ which produce the morphisms r^n_X and $r^n_{X_{d-2}}$ respectively.

Let $x \in X \setminus X_{d-2}$. A small neighbourhood of x, U_x, verifies property (3) if there exist a natural number n_0 such that U_x does not intersect $T^{n_0} X_{d-r}$ for any r. Moreover, this number n_0 exists if and only if $U_x \cap X_{d-2}$ is empty.

Let $x \in X_{d-r} \setminus X_{d-r-1}$, a small neighbourhood of x, U_x, verifies property (4) if there exist a natural number n_0 such that U_x does not intersect $T^{n_0} X_{d-k}$ for any $k > r$ while the intersection $U_x \cap (\overline{TX_{d-r}} \setminus T^{n_0} X_{d-r})$ is not empty.

The following figure shows U_x where $x \in X_{d-m} \setminus X_{d-m-1}$ in Figure 4.9.

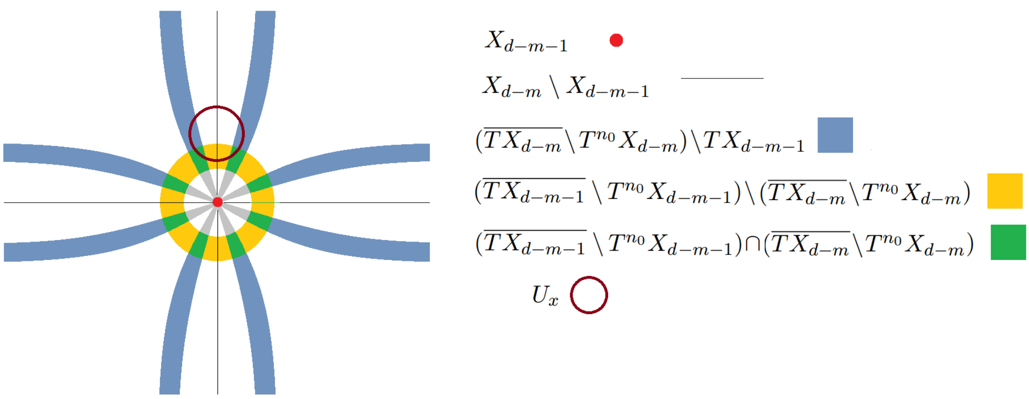

Figure 4.10

□

Definition 4.3.3. Let x be any point of X. If $x \in X \setminus X_{d-2}$, a *principal neighbourhood* of x is a small neighbourhood which verifies property 4.3.2 (3). If $x \in X_{d-2}$, a *principal neighbourhood* of x is a small neighbourhood which verifies property 4.3.2 (4).

Definition 4.3.4. Let $x \in X_{d-r} \setminus X_{d-r-1}$. A *carved principal neighbourhood* of x is an open subset U_x^* equal to $U_x \setminus X_{d-r}$ where U_x is a principal neighbourhoods of x.

Analogously to property 4.3.2 (4) we have

Proposition 4.3.5. *If U_x^* is a carved principal neighbourhood of $x \in X_{d-r} \setminus X_{d-r-1}$, there exist $n_0 \in \mathbb{N}$ such that, for every $n > n_0$, the diagram (4.1) restricts to the diagram*

$$\begin{tikzcd} I^{\bar{p},n}(X_{d-2}) \cap \pi^{-1}(U_x^*) \arrow[r, "\nu^n"] \arrow[d, shift left, "i^n_{X_{d-2}}"] & I^{\bar{p},n}X \cap \pi^{-1}(U_x^*) \arrow[d, shift right, "i^n_X"'] \\ I^{\bar{p},n+1}(X_{d-2}) \cap \pi^{-1}(U_x^*) \arrow[r, "\nu^{n+1}"] \arrow[u, shift left, "r^n_{X_{d-2}}"] & I^{\bar{p},n+1}X \cap \pi^{-1}(U_x^*) \arrow[u, shift right, "r^n_X"'] \end{tikzcd} \tag{4.3}$$

Proposition 4.3.6. *If U_x is a principal neighbourhood of $x \in X_{d-r} \setminus X_{d-r-1}$ for $r > 0$, and $n \in \mathbb{N}$ is big enough, the cohomology group*

$$H^i(I^{\bar{p},n}X \cap \pi^{-1}(U_x), I^{\bar{p},n}X_{d-2} \cap \pi^{-1}(U_x); \mathbb{Q})$$

is 0 if $i \leq \bar{q}(r)$ and isomorphic to the i-th cohomology of the pair $(\sigma^{\partial}_{d-r})^{-1}(x) \subset (I^{\bar{p}}_{r-1}X, I^{\bar{p}}_{r-1}(X_{d-2}))$ if $i > \bar{q}(r)$.

Proposition 4.3.7. *If U_x^* is a carved principal neighbourhood of $x \in X_{d-r} \setminus X_{d-r-1}$ for $r > 0$, and $n \in \mathbb{N}$ is big enough, the cohomology group*

$$H^i(I^{\bar{p},n}X \cap \pi^{-1}(U_x), I^{\bar{p},n}X_{d-2} \cap \pi^{-1}(U_x); \mathbb{Q})$$

is isomorphic to the i-th cohomology of the pair $(\sigma^{\partial}_{d-r})^{-1}(x) \subset (I^{\bar{p}}_{r-1}X, I^{\bar{p}}_{r-1}(X_{d-2}))$ for every $i \in \mathbb{Z}$.

4.4 Sheafification

4.4.1 Sheaf of cubical singular cochains

In this section, every topological space is hereditally paracompact and locally contractible. In particular, the topological pseudomanifold and the intersection spaces of the previous section verify these properties.

In order to produce constructible complexes whose hypercohomology compute the cohomology of intersection spaces we use sheaves of singular cohomology cochains. For technical reasons cubical cochains, as developed by Massey in [Mas, Chapters 7 and 12], adapt best to our construction. Here we sketch very briefly the main points we need; the reader should check [Mas] for complete definitions and proofs.

We denote by $(C_\bullet(X,\mathbb{Q}),\partial)$ the complex of cubical chains of a space X. The group $C_i(X,\mathbb{Q})$ is defined to be the quotient

$$C_i(X,\mathbb{Q}) := Q_i(X,\mathbb{Q})/D_i(X,\mathbb{Q}),$$

where $Q_i(X,\mathbb{Q})$ is the vector space spanned by maps from the i-cube to X and $D_i(X,\mathbb{Q})$ is the subspace of degenerate maps (maps which are constant in one direction of the cube). The differential ∂ is defined in the usual way. The functor given by the homology of the complex $(C_\bullet(\textbf{.},\mathbb{Q}),\partial)$ defines a homology theory with coefficients in $\mathbb{Q}$.

Let $(C^\bullet(X,\mathbb{Q}),\delta)$ be the complex of cubical cochains of X. It is by definition the dual of $(C_\bullet(X,\mathbb{Q}),\partial)$, and hence $C^i(X,\mathbb{Q})$ is the subspace of $Hom(Q_i(X,\mathbb{Q}),\mathbb{Q})$ formed by elements vanishing at $D_i(X,\mathbb{Q})$. The functor given by the cohomology of the complex $(C^\bullet(\textbf{.},\mathbb{Q}),\partial)$ defines a cohomology theory with coefficients in $\mathbb{Q}$.

Let $f: X \to Y$ be a continuous map. We denote by

$$f_{\#\, i} : C_i(X,\mathbb{Q}) \to C_i(Y,\mathbb{Q}),$$
$$f^{\#\, i} : C^i(Y,\mathbb{Q}) \to C^i(X,\mathbb{Q})$$

the associated transformations of complexes of cubical chains and cochains. They form morphisms of complexes

$$f_\# : (C_\bullet(X,\mathbb{Q}),\partial) \to (C_\bullet(Y,\mathbb{Q}),\partial),$$
$$f^\# : (C^\bullet(Y,\mathbb{Q}),\delta) \to (C^\bullet(X,\mathbb{Q}),\delta).$$

Let $f,g : X \to X$ two continuous maps. If f and g are homotopic, then $f_\#$ and $g_\#$ are homotopic morphism of complexes, and the same happens for $f^\#$ and $g^\#$. We need for later use an explicit form of a homotopy of complexes between $f^\#$ and $g^\#$. Let

$$\rho : C_i(X,\mathbb{Q}) \to C_{i+1}(I\times X,\mathbb{Q})$$

be the morphism such that, if σ_i is a singular i-cube in X, $\rho(\sigma_i) = Id_I \times \sigma_i$ (the homomorphism ρ takes degenerate cubical chains to degenerate cubical chains). Let $H: I\times X \to Y$ be an homotopy between f and g, that is, $H_0 = f$ and $H_1 = g$. A homotopy between the morphism of complexes $f_\#$ and $g_\#$ is given by $H_\# \circ \rho$. The dual morphism of $H_\# \circ \rho$ is an homotopy between $f^\#$ and $g^\#$.

Lemma 4.4.1. *Let $h: Z \to X$ a continuous map. If $(H_t)_{|\mathrm{Im}(h)}$ is independent of $t \in I$, then for every $\sigma_i \in Q_i(Z,\mathbb{Q})$, $H_\# \circ \rho \circ h_\#(\sigma_i)$ is degenerate.*

Now we produce a sheafification of cubical cochains. This is an adaptation of the sheafification of singular chains appearing in [Ram].

Definition 4.4.2. For every $i \in \mathbb{Z}_{\geq 0}$, let C^i be the presheaf of vector spaces

$$U \rightsquigarrow C^i(U,\mathbb{Q})$$

where the restriction morphisms are the obvious ones.

The *sheaf of cubical singular i-cochains* $\mathcal{C}^i_X$ is defined to be the sheafification of C^i.

For every $i \in \mathbb{Z}_{\geq 0}$, let $C^i_\circ(X)$ be the vector subspace of $C^i(X)$ given by the set of cochains $\xi^i \in C^i(X)$ such that there exist an open covering $\{U_j\}_{j\in J}$ of X such that $\xi^i_{|U_j} = 0$ for every $j \in J$. As in [Ram] one shows that the sheafification is defined by

$$\mathcal{C}^i_X(U) = C^i(U)/C^i_\circ(U).$$

At the level of sheaves we have functoriality as well. Let $f : X \to Y$ be a continuous map. Then, f induces a morphism of complexes of sheaves on Y

$$f^\# : \mathcal{C}^\bullet_Y \to f_*\mathcal{C}^\bullet_X.$$

As one expects, if X is contractible then

$$H^i(\mathcal{C}^i(X)) \cong \begin{cases} \mathbb{Q} & \text{if} \quad i = 0 \\ 0 & \text{if} \quad i \neq 0 \end{cases}$$

This implies that the complex of sheaves $\mathcal{C}^\bullet_X$ is a resolution of the constant sheaf $\mathbb{Q}_X$.

Moreover, for every $i \in \mathbb{Z}_{\geq 0}$, the sheaf $\mathcal{C}^i$ is flabby. Indeed, it is enough to prove the restriction morphisms of the presheaf, $C^i(X,\mathbb{Q}) \to C^i(U,\mathbb{Q})$, are surjective for every open subset $U \subset X$. Given $\xi \in C^i(U,\mathbb{Q})$, let $\xi_X \in C^i(X,\mathbb{Q})$ be the linear morphism $C_i(X,\mathbb{Q}) \to \mathbb{Q}$ such that, for every singular i-cube σ in X, we have

$$\xi_X(\sigma) = \begin{cases} \xi(\sigma) & \text{if} \quad \mathrm{Im}(\sigma) \subset U \\ 0 & \text{if} \quad \mathrm{Im}(\sigma) \not\subset U \end{cases}$$

Then, $(\xi_X)_{|U} = \xi$.

Corollary 4.4.3. *For every $i \in \mathbb{Z}_{\geq 0}$, the i-th cohomology group $H^i(X,\mathbb{Q})$ is isomorphic to i-th group of cohomology of the complex $\mathcal{C}^\bullet_X(X)$.*

4.4.2 The intersection space constructible complex

Let X be a topological pseudomanifold with stratification:

$$X = X_d \supset X_{d-2} \supset \dots \supset X_0 \supset X_{-1} = \emptyset$$

and a conical structure given by the stratification such that there exist a set of choices so that the inductive construction of the intersection space of X is not obstructed. Let X' be the homotopy model of X and $\pi : X' \to X$ the homotopy equivalence. Let $(I^{\bar{p},n}X, I^{\bar{p},n}(X_{d-2}))$ with $n \in \mathbb{N}$ be the associated sequence of intersection space pairs and

$$j^n : I^{\bar{p},n}X \to X'$$
$$\mu^n : I^{\bar{p},n}(X_{d-2}) \to X'$$

the canonical inclusions.

In order to lighten the formulas appearing in this section we denote by $\mathcal{C}^{n,\bullet}_X$ and $\mathcal{C}^{n,\bullet}_{X_{d-2}}$ the complex of sheaves of cubical singular chains in $I^{\bar{p},n}X$ and $I^{\bar{p},n}(X_{d-2})$ respectively.

Proposition 4.4.4. *For every $n \in \mathbb{N}$, there exist a commutative diagram*

$$\begin{tikzcd} j_*^{n+1}\mathcal{C}_X^{n+1,\bullet} \arrow[r, "\nu^{n+1\#}"] \arrow[d, "i_X^{n\#}"] & \mu_*^{n+1}\mathcal{C}_{X_{d-2}}^{n+1,\bullet} \arrow[d, "i_{X_{d-2}}^{n\#}"] \\ j_*^n\mathcal{C}_X^{n,\bullet} \arrow[r, "\nu^{n\#}"] & \mu_*^n\mathcal{C}_{X_{d-2}}^{n,\bullet} \end{tikzcd} \tag{4.4}$$

where all the morphisms are surjective.

Proof. For every open subset $U \subset X'$, the inclusions of Diagram (4.1) restrict to a diagram

$$\begin{tikzcd} I^{\bar{p},n}(X_{d-2}) \cap U \arrow[r, "\nu^n"] \arrow[d, "i_{X_{d-2}}^n"] & I^{\bar{p},n}X \cap U \arrow[d, "i_X^n"'] \\ I^{\bar{p},n+1}(X_{d-2}) \cap U \arrow[r, "\nu^{n+1}"] & I^{\bar{p},n+1}X \cap U \end{tikzcd}$$

So, we have the following diagram between the cubical cochain groups:

$$\begin{tikzcd} C^i(I^{\bar{p},n+1}X \cap U, \mathbb{Q}) \arrow[r, "\nu^{n+1\#}"] \arrow[d, "i_X^{n\#}"] & C^i(I^{\bar{p},n+1}(X_{d-2}) \cap U, \mathbb{Q}) \arrow[d, "i_{X_{d-2}}^{n\#}"] \\ C^i(I^{\bar{p},n}X \cap U, \mathbb{Q}) \arrow[r, "\nu^{n\#}"] & C^i(I^{\bar{p},n}(X_{d-2}) \cap U, \mathbb{Q}) \end{tikzcd}$$

The morphisms of this diagram induce the morphisms of the proposition.

Moreover, these morphisms are surjective since every inclusion of topological spaces induces a surjection between the corresponding cubical cochain groups. $\square$

Denote by $\mathcal{K}^{n,\bullet}$ the kernel of $\nu^{n\#}$. There is a canonical morphism

$$i^{n\#} : \mathcal{K}^{n+1,\bullet} \to \mathcal{K}^{n,\bullet}.$$

Remark 4.4.5. For every $i \in \mathbb{Z}_{\geq 0}$ and every $n \in \mathbb{N}$, the i-th rational cohomology group of the pair $(I^{\bar{p},n}X, I^{\bar{p},n}(X_{d-2}))$ is isomorphic to i-th cohomology group of the complex $\mathcal{K}^{n,\bullet}$.

Definition 4.4.6. Given a pair of natural numbers $n_1 < n_2$, we will define

$$i^{n_1,n_2} := i^{n_1\#} \circ ... \circ i^{n_2-1\#} : \mathcal{K}^{n_2,\bullet} \to \mathcal{K}^{n_1,\bullet}$$

Then, the complexes of sheaves $\{\mathcal{K}^{n,\bullet}\}_{n\in\mathbb{N}}$ and the morphisms i^{n_1,n_2} form an inverse system and we can consider the inverse limit

$$\varprojlim_{n\in\mathbb{N}} \mathcal{K}^{n,\bullet},$$

which is a complex of sheaves.

Lemma 4.4.7. *$(\pi_* \varprojlim_{n\in\mathbb{N}} \mathcal{K}^{n,\bullet})_{X\setminus X_{d-2}}$ is quasi-isomorphic to $\mathbb{Q}_{X\setminus X_{d-2}}$.*

Proof. Let $x \in X \setminus X_{d-2}$. We have the following equalities:

$$(\pi_* \varprojlim_{n\in\mathbb{N}} \mathcal{K}^{n,\bullet})_x = \varinjlim_{x\in U \text{open}} \pi_*(\varprojlim_{n\in\mathbb{N}} \mathcal{K}^{n,\bullet})(U) = \varinjlim_{x\in U \text{open}} (\varprojlim_{n\in\mathbb{N}} \mathcal{K}^{n,\bullet})(\pi^{-1}(U)) =$$

$$= \varinjlim_{x\in U \text{open}} \varprojlim_{n\in\mathbb{N}} (\mathcal{K}^{n,\bullet}(\pi^{-1}(U))).$$

Let U_x be a principal neighbourhood of x (see definition 4.3.3). There exist $n_0 \in \mathbb{N}$ such that, for every open subset $U \subset U_x$ and for every $n > n_0$, we have

$$\pi^{-1}(U) \cap I^{\bar{p},n} X = \pi^{-1}(U)$$

and

$$\pi^{-1}(U) \cap I^{\bar{p},n}(X_{d-2}) = \emptyset$$

Consequently, $\mathcal{K}^{n,\bullet}(\pi^{-1}(U)) = j^n_* \mathcal{C}^{n,\bullet}_X(\pi^{-1}(U)) = \mathcal{C}^\bullet_{X'}(\pi^{-1}(U))$ where $\mathcal{C}_{X'}$ is the sheaf of singular i-cochains in X'.

Moreover, for every $n > n_0$, $i^{n\#} = i^{n\#}_X = Id_{\mathcal{C}^\bullet_{X'}(\pi^{-1}(U))}$. So, $\varprojlim_{n\in\mathbb{N}}(\mathcal{K}^{n,\bullet}(\pi^{-1}(U))) = \mathcal{C}^\bullet_{X'}(\pi^{-1}(U))$.

Thus, we have shown that $(\pi_* \varprojlim_{n\in\mathbb{N}} \mathcal{K}^{n,\bullet})_{X\setminus X_{d-2}}$ is quasi-isomorphic to $\pi_*\mathbb{Q}_{\pi^{-1}(X\setminus X_{d-2})}$, and the later sheaf is quasi-isomorphic to $\mathbb{Q}_{X\setminus X_{d-2}}$ since $\pi|_{\pi^{-1}(X\setminus X_{d-2})}$ is a homotopy equivalence. ☐

We study now the cohomology of the complex $\pi_* \varprojlim_{n\in\mathbb{N}} \mathcal{K}^{n,\bullet}$ over each of the deeper strata of X and over the global sections. With this purpose, we study the cohomology of $\pi_* \varprojlim_{n\in\mathbb{N}} \mathcal{K}^{n,\bullet}$ in the principal and the carved principal neighbourhoods (see definitions 4.3.3 and 4.3.4) and in the total space.

Proposition 4.4.8. *Let U be equal to X or a principal neighbourhood or a carved principal neighbourhood of some $x \in X_{d-k} \setminus X_{d-(k+1)}$. Then,*

$$H^i(\varprojlim_{n\in\mathbb{N}}(\mathcal{K}^{n,\bullet}(\pi^{-1}(U)))) \cong \varprojlim_{n\in\mathbb{N}} H^i(\mathcal{K}^{n,\bullet}(\pi^{-1}(U)))$$

Now, we need some preliminary work in order to prove Proposition 4.4.8.

Lemma 4.4.9. *Let U be equal to X or a principal neighbourhood or a carved principal neighbourhood of some $x \in X_{d-k} \setminus X_{d-(k+1)}$. Then, there exist $n_0 \in \mathbb{N}$ such that if $n > n_0$, there exist a morphism*

$$r^{n\#}_U : \mathcal{K}^{n,\bullet}(\pi^{-1}(U)) \to \mathcal{K}^{n+1,\bullet}(\pi^{-1}(U))$$

such that $i^{n\#}(\pi^{-1}(U)) \circ r^{n\#}_U = Id_{\mathcal{K}^{n,\bullet}(\pi^{-1}(U))}$ and $r^{n\#}_U \circ i^{n\#}(\pi^{-1}(U))$ is homotopic to the identity.

Moreover, there exists a homotopy h^n_U between $r^{n\#}_U \circ i^{n,\#}(\pi^{-1}(U))$ and $Id_{\mathcal{K}^{n+1,\bullet}(\pi^{-1}(U))}$ such that $i^{n\#}(\pi^{-1}(U)) \circ h^n_U = 0$.

Proof. Let U be a principal neighbourhood and let n_0 be the natural number of Proposition 4.3.2 (4). For every $n > n_0$, the diagram (4.2) induces a diagram

$$\begin{tikzcd} j_*^{n+1}\mathcal{C}_X^{n+1,\bullet}(\pi^{-1}(U)) \arrow[r, "\nu^{n+1\#}"] \arrow[d, shift left, "i_X^{n\#}"] & \mu_*^{n+1}\mathcal{C}_{X_{d-2}}^{n+1,\bullet}(\pi^{-1}(U)) \arrow[d, shift right, "i_{X_{d-2}}^{n\#}"'] \\ j_*^{n}\mathcal{C}_X^{n,\bullet}(\pi^{-1}(U)) \arrow[r, "\nu^{n\#}"] \arrow[u, shift left, "r_X^{n\#}"] & \mu_*^{n}\mathcal{C}_{X_{d-2}}^{n,\bullet}(\pi^{-1}(U)) \arrow[u, shift right, "r_{X_{d-2}}^{n\#}"'] \end{tikzcd}$$

where $i_{X_{d-2}}^{n\#} \circ r_{X_{d-2}}^{n\#} = Id_{\mu_*^n\mathcal{C}_{X_{d-2}}^{n,\bullet}(\pi^{-1}(U))}$, $i_X^{n\#} \circ r_X^{n\#} = Id_{j_*^n\mathcal{C}_X^{n,\bullet}(\pi^{-1}(U))}$ and $r_{X_{d-2}}^{n\#} \circ i_{X_{d-2}}^{n\#}$ and $r_X^{n\#} \circ i_X^{n\#}$ are homotopic to the identity.

Then, we obtain a canonical morphism

$$r_U^{n\#} : \mathcal{K}^{n,\bullet}(\pi^{-1}(U)) \to \mathcal{K}^{n+1,\bullet}(\pi^{-1}(U))$$

such that $i^{n\#}(\pi^{-1}(U)) \circ r_U^{n\#} = Id_{\mathcal{K}^{n,\bullet}(\pi^{-1}(U))}$ and $r_U^{n\#} \circ i^{n\#}(\pi^{-1}(U))$ is homotopic to the identity.

Consider the diagram (4.2) of Proposition 4.3.2 (4). There exist homotopies

$$H_X^n : \pi^{-1}(U) \cap I^{\bar{p},n+1}X \times I \to \pi^{-1}(U) \cap I^{\bar{p},n+1}X$$

between $i_X^n \circ r_X^n$ and $Id_{\pi^{-1}(U)\cap I^{\bar{p},n+1}X}$ and

$$H_{X_{d-2}}^n : \pi^{-1}(U) \cap I^{\bar{p},n+1}(X_{d-2}) \times I \to \pi^{-1}(U) \cap I^{\bar{p},n+1}(X_{d-2})$$

between $i_{X_{d-2}}^n \circ r_{X_{d-2}}^n$ and $Id_{\pi^{-1}(U)\cap I^{\bar{p},n+1}(X_{d-2})}$. Moreover, we can suppose that, for every $t \in I$, the restrictions of H_{Xt}^n and $H_{X_{d-2}t}^n$ to $\pi^{-1}(U) \cap I^{\bar{p},n}X$ and $\pi^{-1}(U) \cap I^{\bar{p},n}(X_{d-2})$ are the identity respectively.

Following the procedure explained in Section 4.4.1 the mapping H_X^n induces a homotopy between $i_X^{n\#} \circ r_X^{n\#}$ and $Id_{j_*^{n+1}\mathcal{C}_X^{n+1,\bullet}(\pi^{-1}(U))}$. Moreover, applying Lemma 4.4.1 we have that $i_X^{n\#} \circ h_X^n$ is equal to 0.

Similarly the mapping $H_{X_{d-2}}^n$ induce a homotopy $h_{X_{d-2}}^n$ between $i_{X_{d-2}}^{n\#} \circ r_{X_{d-2}}^{n\#}$ and $Id_{\mu_*^{n+1}\mathcal{C}_{X_{d-2}}^{n+1,\bullet}(\pi^{-1}(U))}$ such that $i_{X_{d-2}}^{n\#} \circ h_{X_{d-2}}^n$ is equal to 0.

So, there exist a homotopy h_U^n between $i^{n\#}(\pi^{-1}(U)) \circ r_U^{n\#}$ and $Id_{\mathcal{K}^{n+1,\bullet}(\pi^{-1}(U))}$ such that $i^{n\#}(\pi^{-1}(U)) \circ h_U^n = 0$.

If U is equal to X or a carved principal neighbourhood we can apply the same method using the diagram (4.1) of Proposition 4.3.2 or the diagram of Proposition 4.3.5, respectively. □

Remark 4.4.10. The diagram (4.1) of Proposition 4.3.2 is valid for every $n \in \mathbb{N}$. So, in the previous lemma, we can take $n_0 = 0$ if U is the total space.

Remark 4.4.11. Note that the morphisms $r_U^{n\#}$ do not induce a morphism of complexes of sheaves since they are not defined for every open subset.

Definition 4.4.12. Given a pair of natural numbers $n_1 < n_2$ such that $n_0 < n_1$, we define

$$r_U^{n_1,n_2} := r_U^{n_2-1\#} \circ ... \circ r_U^{n_1\#} : \mathcal{K}^{n_1,\bullet}(\pi^{-1}(U)) \to \mathcal{K}^{n_2,\bullet}(\pi^{-1}(U))$$

Remark 4.4.13. For every $n > n_0$, the inclusions $\pi^{-1}(U) \cap I^{\bar{p},n}X \hookrightarrow \pi^{-1}(U) \cap I^{\bar{p},n+1}X$ and $\pi^{-1}(U) \cap I^{\bar{p},n}X_{d-2} \hookrightarrow \pi^{-1}(U) \cap I^{\bar{p},n+1}X_{d-2}$ are homotopy equivalences. So, $i_X^{n\#}(\pi^{-1}(U))$ and $i_{X_{d-2}}^{n\#}(\pi^{-1}(U))$ are quasi-isomorphisms.

Then,

$$i^{n\#}(\pi^{-1}(U)) : \mathcal{K}^{n+1,\bullet}(\pi^{-1}(U))) \to \mathcal{K}^{n,\bullet}(\pi^{-1}(U)))$$

is also a quasi-isomorphism and we have an isomorphism

$$\begin{array}{ccc} \varprojlim_{n\in\mathbb{N}} H^i(\mathcal{K}^{n,\bullet}(\pi^{-1}(U))) & \longrightarrow & H^i(\mathcal{K}^{n_0+1,\bullet}((\pi^{-1}(U)))) \\ \{[\xi_n^i]\}_{n\in\mathbb{N}} & \longrightarrow & [\xi_{n_0+1}^i] \end{array}$$

Notation 4.4.14. For every open subset $V \subset X'$, the elements of $H^i(\varprojlim_{n\in\mathbb{N}} \mathcal{K}^{n,\bullet}(V))$ are equivalence classes of elements

$$\{\xi_n^i\}_{n\in\mathbb{N}} \in \mathrm{Ker}(\varprojlim_{n\in\mathbb{N}} \partial_n^{i+1}(V)) \subset \varprojlim_{n\in\mathbb{N}} \mathcal{K}^{n,i}(V)$$

which we are going to denote with $[\{\xi_n^i\}_{n\in\mathbb{N}}]$.

In addition, given $n \in \mathbb{N}$ and an element $\xi_n^i \in \mathrm{Ker}(\partial_n^{i+1}(V)) \subset \mathcal{K}^{n,i}(V)$, we are going to denote its equivalence class in $H^i(\mathcal{K}^{n,\bullet}(V))$ with $[\xi_n^i]$.

Proof of Proposition 4.4.8. It is enough to prove that, if U is equal to X or a principal neighbourhood or a carved principal neighbourhood of x, then the morphism

$$\begin{array}{ccc} \mathrm{Ker}(\varprojlim_{n\in\mathbb{N}} \partial_n^{i+1}(\pi^{-1}(U))) & \xrightarrow{\alpha} & \varprojlim_{n\in\mathbb{N}} H^i(\mathcal{K}^{n,\bullet}(\pi^{-1}(U))) \\ \{\xi_n^i\}_{n\in\mathbb{N}} & \longrightarrow & \{[\xi_n^i]\}_{n\in\mathbb{N}} \end{array}$$

factorices into a morphism

$$H^i(\varprojlim_{n\in\mathbb{N}} \mathcal{K}^{n,\bullet}(\pi^{-1}(U))) \xrightarrow{\beta} \varprojlim_{n\in\mathbb{N}} H^i(\mathcal{K}^{n,\bullet}(\pi^{-1}(U)))$$

which is an isomorphism.

First, we prove α factorices. Let us consider an element $\{\xi_n^i\}_{n\in\mathbb{N}} \in \mathrm{Im}(\varprojlim_{n\in\mathbb{N}} \partial_n^i(\pi^{-1}(U)))$. Then, there exist an element $\{\delta_n^{i-1}\}_{n\in\mathbb{N}} \in \varprojlim_{n\in\mathbb{N}} \mathcal{K}^{n,i-1}(\pi^{-1}(U))$ such that

$$\varprojlim_{n\in\mathbb{N}} \partial_n^i(\pi^{-1}(U))(\{\delta_n^{i-1}\}_{n\in\mathbb{N}}) = \{\xi_n^i\}_{n\in\mathbb{N}}.$$

So, for every $n \in \mathbb{N}$, $\partial_n^i(\pi^{-1}(U))(\delta_n^{i-1}) = \xi_n^i$. Consequently, $\alpha(\{\xi_n^i\}_{n\in\mathbb{N}}) = \{[\xi_n^i]\}_{n\in\mathbb{N}} = 0$ and we conclude that the morphism factorices.

Now, we prove that α and, consequently, β are surjective.

Because of Remark 4.4.13, it is enough to prove that, for every element $\xi \in \mathrm{Ker}(\partial^{i+1}_{n_0+1}((\pi^{-1}(U))))$, there exist an element $\{\xi^i_n\}_{n\in\mathbb{N}} \in \mathrm{Ker}(\varprojlim_{n\in\mathbb{N}} \partial^{i+1}_n(\pi^{-1}(U)))$ such that $\xi^i_{n_0+1} = \xi$.

Let us consider

$$\xi^i_n = \begin{cases} (i^{n,n_0+1}(\pi^{-1}(U)))(\xi) & \text{if} \quad n < n_0+1 \\ \xi & \text{if} \quad n = n_0+1 \\ r_U^{n_0+1,n}(\xi) & \text{if} \quad n > n_0+1 \end{cases}$$

Then, for every pair of natural numbers $n_1 < n_2$, we have the equality

$$((i^{n_1,n_2})(\pi^{-1}(U)))(\xi^i_{n_2}) = \xi^i_{n_1}.$$

Hence $\{\xi^i_n\}_{n\in\mathbb{N}}$ belongs to $\varprojlim_{n\in\mathbb{N}} \mathcal{K}^{n,i}(\pi^{-1}(U))$.

Moreover

- we have the vanishing $(\partial^{i+1}_{n_0+1}(\pi^{-1}(U)))(\xi) = 0$,
- for every pair of natural numbers $n_1 < n_2$ we have the equality

$$\partial^i_{n_1} \circ i^{n_1,n_2} = i^{n_1,n_2} \circ \partial^i_{n_2}$$

  and,
- if $n_1 > n_0$, we have the equality

$$\partial^i_{n_2}(\pi^{-1}(U)) \circ r_U^{n_1,n_2} = r_U^{n_1,n_2} \circ \partial^i_{n_1}(\pi^{-1}(U)).$$

Consequently $(\partial^{i+1}_n(\pi^{-1}(U)))(\xi^i_n) = 0$ for every $n \in \mathbb{N}$, $\{\xi^i_n\}_{n\in\mathbb{N}}$ belongs to $\mathrm{Ker}(\varprojlim_{n\in\mathbb{N}} \partial^{i+1}_n(\pi^{-1}(U)))$ and α and β are surjective.

Finally, we prove β is injective, that is, $\mathrm{Ker}(\alpha) = \mathrm{Im}(\varprojlim_{n\in\mathbb{N}} \partial^i_n(\pi^{-1}(U)))$.

Let $\{\xi^i_n\}_{n\in\mathbb{N}} \in \mathrm{Ker}(\varprojlim_{n\in\mathbb{N}} \partial^{i+1}_n)(\pi^{-1}(U))$ such that $\alpha(\{\xi^i_n\}_{n\in\mathbb{N}}) = \{[\xi^i_n]\}_{n\in\mathbb{N}} = 0$. Then, $\xi^i_{n_0+1} \in \mathrm{Im}(\partial^i_{n_0+1}(\pi^{-1}(U)))$. So, there exist $\delta \in \mathcal{K}^{n_0+1,\,i-1}(\pi^{-1}(U))$ such that $(\partial^i_{n_0+1}(\pi^{-1}(U)))(\delta) = \xi^i_{n_0+1}$.

For every $n \in \mathbb{N}$, we define

$$\delta^{i-1}_n = \begin{cases} (i^{n,n_0+1}(\pi^{-1}(U)))(\delta) & \text{if} \quad n < n_0+1 \\ \delta & \text{if} \quad n = n_0+1 \\ r_U^{n_0+1,n}(\delta) & \text{if} \quad n > n_0+1 \end{cases}$$

Then, $\{\delta^{i-1}_n\}_{n\in\mathbb{N}} \in \varprojlim_{n\in\mathbb{N}} \mathcal{K}^{n,\,i-1}(\pi^{-1}(U))$.

Let us denote $\widetilde{\xi^i_n} := (\partial^i_n(\pi^{-1}(U)))(\delta^{i-1}_n)$ for every $n \in \mathbb{N}$. Then we have the equality $(\varprojlim_{n\in\mathbb{N}} \partial^i_n(\pi^{-1}(U)))(\{\delta^{i-1}_n\}_{n\in\mathbb{N}}) = \{\widetilde{\xi^i_n}\}_{n\in\mathbb{N}}$ and $[\{\widetilde{\xi^i_n}\}_{n\in\mathbb{N}}] = 0$ in $H^i(\varprojlim_{n\in\mathbb{N}} \mathcal{K}^{n,\bullet}(\pi^{-1}(U)))$.

So, to prove β is injective, it is enough to prove the equality $[\{\widetilde{\xi^i_n}\}_{n\in\mathbb{N}}] = [\{\xi^i_n\}_{n\in\mathbb{N}}]$ in $H^i(\varprojlim_{n\in\mathbb{N}} \mathcal{K}^{n,\bullet}(\pi^{-1}(U)))$.

If $n < n_0+1$, we have

$$\widetilde{\xi^i_n} = \partial^i_n(\pi^{-1}(U))(\delta^{i-1}_n) = \partial^i_n(\pi^{-1}(U))(i^{n,n_0+1}(\pi^{-1}(U)))(\delta)) =$$
$$= i^{n,n_0+1}(\pi^{-1}(U))(\partial^i_{n_0+1}(\pi^{-1}(U))(\delta)) = i^{n,n_0+1}(\pi^{-1}(U))(\xi^i_{n_0+1}) = \xi^i_n.$$

If $n = n_0+1$, we have

$$\widetilde{\xi^i_{n_0+1}} = \partial^i_{n_0+1}(\pi^{-1}(U))(\delta) = \xi^i_{n_0+1}.$$

If $n > n_0+1$, we have

$$\widetilde{\xi^i_n} = \partial^i_n(\pi^{-1}(U))(\delta^{i-1}_n) = \partial^i_n(\pi^{-1}(U))(r^{n_0+1,n}_U(\delta)) = r^{n_0+1,n}_U(\partial^i_{n_0+1}(\pi^{-1}(U))(\delta)) =$$
$$= r^{n_0+1,n}_U(\xi^i_{n_0+1}) = r^{n_0+1,n}_U(i^{n_0+1,n}(\pi^{-1}(U))(\xi^i_n)).$$

For every $n > n_0+1$, let h^n_U be the homotopy defined in Lemma 4.4.9. A simple computation shows that, for every $n > n_0+1$, we have the equality:

$$\widetilde{\xi^i_n} - \xi^i_n = \partial^i_n(\sum_{k=n_0+1}^{n-1} (r^{k,n}_U \circ h^{k-1}_U)(\xi^i_k) + h^{n-1}_U(\xi^i_n)).$$

Let

$$\epsilon^{i-1}_n = \begin{cases} 0 & \text{if} \quad n \le n_0+1 \\ \sum_{k=n_0+2}^{n-1}(r^{k,n}_U \circ h^{k-1}_U)(\xi^i_k) + h^{n-1}_U(\xi^i_n) & \text{if} \quad n > n_0+1 \end{cases}$$

Let us prove $\{\epsilon^{i-1}_n\}_{n\in\mathbb{N}} \in \varprojlim_{n\in\mathbb{N}}(\mathcal{K}^{n,\,i-1}(\pi^{-1}(U)))$.

Since, for every $n > n_0+1$, $i^{n\#}(\pi^{-1}(U)) \circ h^n_U = 0$ and $i^{n\#}(\pi^{-1}(U)) \circ r^{n\#}_U = Id_{\mathcal{K}^{n,\bullet}(\pi^{-1}(U))}$, if $n_1 > n_0+1$, we have the equality

$$i^{n_1,n_2}(\pi^{-1}(U))(\sum_{k=n_0+2}^{n_2-1} (r^{k,n_2}_U \circ h^{k-1})(\xi^i_k) + h^{n_2-1}_U(\xi^i_{n_2})) = \sum_{k=n_0+2}^{n_1-1} (r^{k,n_1}_U \circ h^{k-1}_U)(\xi^i_k) +$$
$$+ h^{n_1-1}_U(\xi^i_{n_1}),$$

and if $n_1 \le n_0+1$, we have

$$i^{n_1,n_2}(\pi^{-1}(U))(\sum_{k=n_0+2}^{n_2-1} (r^{k,n_2}_U \circ h^{k-1}_U)(\xi^i_k) + h^{n_2-1}_U(\xi^i_{n_2})) = 0.$$

Then, $\{\epsilon^{i-1}_n\}_{n\in\mathbb{N}}$ belongs to $\varprojlim_{n\in\mathbb{N}}(\mathcal{K}^{n,\,i-1}(\pi^{-1}(U)))$ and we have the equality

$$\varprojlim_{n\in\mathbb{N}} \partial^i(\pi^{-1}(U))(\{\epsilon^{i-1}_n\}_{n\in\mathbb{N}}) = \{\widetilde{\xi^i_n}\}_{n\in\mathbb{N}} - \{\xi^i_n\}_{n\in\mathbb{N}}.$$

Therefore, $[\{\widetilde{\xi^i_n}\}_{n\in\mathbb{N}}]$ equals $[\{\xi^i_n\}_{n\in\mathbb{N}}]$ in $H^i(\varprojlim_{n\in\mathbb{N}}(\mathcal{K}^{n,\,i}(\pi^{-1}(U))))$ and we conclude. $\square$

Definition 4.4.15. Let us define $IS := \pi_* \varprojlim_{n\in\mathbb{N}} \mathcal{K}^{n,\bullet}$.

Theorem 4.4.16. *The hypercohomlogy of IS is isomorphic to the cohomology of the intersection space pair.*

Proof. The sheaves $\mathcal{K}^{n,i}$ are flabby. Let U be any open subset in X'. Every section in $\mathcal{K}^{n,i}(U)$, ξ, is also a section of $j^n_*\mathcal{C}^{n,i}(U)$. Then, ξ is the equivalence class of a singular cubical i-cochain $\xi' \in C^i(U \cap I^{\bar{p},n}X, \mathbb{Q})$. We can extend ξ' by 0 to get a singular cubical i-cochain $\xi'_{I^{\bar{p},n}X} \in C^i(I^{\bar{p},n}X, \mathbb{Q})$, that is, for every singular i-cube σ in $I^{\bar{p},n}X$, we have

$$\xi'_{I^{\bar{p},n}X}(\sigma) = \begin{cases} \xi'(\sigma) & \text{if } \operatorname{Im}(\sigma) \subset U \\ 0 & \text{if } \operatorname{Im}(\sigma) \not\subset U \end{cases}$$

The equivalence class of $\xi'_{I^{\bar{p},n}X}$ in the sheaf of singular cubical i-cochains is a section $\xi_{X'}$ in $j^n_*\mathcal{C}^{n,i}(X')$. It is easy to check that $\xi_{X'}$ is contained in $\mathcal{K}^{n,i}(X')$ since it is a extension by 0 of ξ.

Now, we prove $\varprojlim_{n\in\mathbb{N}} \mathcal{K}^{n,i}$ is also flabby. Let $\{\xi_n\}_{n\in\mathbb{N}}$ be a section of $(\varprojlim_{n\in\mathbb{N}} \mathcal{K}^{n,i})(U) = \varprojlim_{n\in\mathbb{N}}(\mathcal{K}^{n,i}(U))$. Since $\mathcal{K}^{n,i}(U)$ is flabby for every natural n, the sections ξ_n extend by 0 to sections $\xi_n^{X'}$ in $\mathcal{K}^{n,i}(X')$. It is easy to check that $i^{n_1,n_2}(X')(\xi_{n_2}^{X'})$ is equal to $\xi_{n_1}^{X'}$ for every pair of natural numbers $n_1 < n_2$. So, $\{\xi_n^{X'}\}_{n\in\mathbb{N}}$ is a global section of $\varprojlim_{n\in\mathbb{N}} \mathcal{K}^{n,i}$.

Then, all the sheaves of the complex IS are flabby and, consequently, the hypercohomology of IS is equal to the cohomology of the global sections of IS. By Proposition 4.4.8, there is an isomorphism

$$H^i(\Gamma(X, IS)) \cong \varprojlim_{n\in\mathbb{N}} H^i(\Gamma(X', \mathcal{K}^{n\bullet})).$$

Since the cohomology $H^i(\Gamma(X', \mathcal{K}^{n\bullet}))$ is the cohomology of the intersection space pair for every $n \in \mathbb{N}$, we conclude. $\square$

Now we prove a set of properties of the IS, in a similar vein that those satisfied by intersection cohomology complexes.

Definition 4.4.17. For $k = 2, ..., d$, we define $U_k := X \setminus X_{d-k}$ and we denote the canonical inclusions with $i_k : U_k \to U_{k+1}$ and $j_k : X_{d-k} \setminus X_{d-(k+1)} \to U_{k+1}$.

Theorem 4.4.18. *The complex of sheaves IS satisfies the following properties.*

(1) $IS_{|U_2}$ is quasi-isomorphic to $\mathbb{Q}_{U_2}$.

(2) The cohomology sheaves $\mathcal{H}^i(IS)$ are 0 if $i \notin \{0, 1, ..., d\}$

(3) For $k = 2, ..., d$, the cohomology sheaves $\mathcal{H}^i(j_k^* IS_{|U_{k+1}})$ are 0 if $i \leq \bar{q}(k)$.

(4) For $k = 2, ..., d$, the usual morphisms between the cohomology sheaves $\mathcal{H}^i(j_k^* IS_{|U_{k+1}}) \to \mathcal{H}^i(j_k^* i_{k*} IS_{|U_k})$ are isomorphisms if $i > \bar{q}(k)$.

Proof. (4.4.18) is shown in lemma 4.4.7.

Let $x \in X_{d-k} \setminus X_{d-(k+1)}$ for some $k \in \{2, 3, ..., d\}$. Given a complex of sheaves we denote by H^i and $\mathcal{H}^i$ its i-th cohomology presheaf and sheaf respectively. We have the obvious chain of equalities:

$$\mathcal{H}^i(\pi_* \varprojlim_{n\in\mathbb{N}} \mathcal{K}^{n,\bullet})_x = H^i(\pi_* \varprojlim_{n\in\mathbb{N}} \mathcal{K}^{n,\bullet})_x = \varinjlim_{x\in U \text{open}} H^i((\varprojlim_{n\in\mathbb{N}} \mathcal{K}^{n,\bullet})(\pi^{-1}(U))) =$$

$$= \varinjlim_{x\in U \text{open}} H^i(\varprojlim_{n\in\mathbb{N}}(\mathcal{K}^{n,\bullet}(\pi^{-1}(U)))).$$

Since the principal neighbourhoods form a system of neighborhoods for any point, we can suppose every open subset U appearing in the previous formula is a principal neighbourhood of x. Then, applying Proposition 4.4.8 we have

$$\mathcal{H}^i(\pi_* \varprojlim_{n\in\mathbb{N}} \mathcal{K}^{n,\bullet})_x = \varinjlim_{\substack{U \text{ principal} \\ \text{neighbourhood of } x}} \varprojlim_{n\in\mathbb{N}} H^i((\mathcal{K}^{n,\bullet}(\pi^{-1}(U)))).$$

So, applying proposition 4.3.6, $\mathcal{H}^i(\pi_* \varprojlim_{n\in\mathbb{N}} \mathcal{K}^{n,\bullet})_x$ is 0 if $i \leq \bar{q}(k)$ and equal to

$$H^i((\sigma^{\partial}_{d-r})^{-1}(x); \mathbb{Q})$$

if $i > \bar{q}(k)$.

Hence, we have proven (4.4.18) and (4.4.18) of the theorem.

Moreover, applying again Proposition 4.4.8

$$\mathcal{H}^i(j_k^* i_{k*} IS_{|U_k})_x = \varinjlim_{\substack{U \text{ principal} \\ \text{neighbourhood of } x}} H^i(\varprojlim_{n\in\mathbb{N}} \pi_* \mathcal{K}^{n,\bullet}(U \setminus X_{d-k})) =$$

$$= \varinjlim_{\substack{U \text{ principal} \\ \text{neighbourhood of } x}} \varprojlim_{n\in\mathbb{N}} H^i(\pi_* \mathcal{K}^{n,\bullet}(U \setminus X_{d-k}))$$

and, because of Proposition 4.3.7,

$$\mathcal{H}^i(j_k^* i_{k*} IS_{|U_k})_x = H^i((\sigma^{\partial}_{d-r})^{-1}(x); \mathbb{Q})$$

for every $i \in \mathbb{Z}$, which concludes the proof. $\square$

4.5 Axioms of Intersection Space Complexes

From now on, we do not need the assumption of Remark 4.2.10.

Let X be a topological pseudomanifold with the following stratification:

$$X = X_d \supset X_{d-2} \supset ... \supset X_0 \supset X_{-1} = \emptyset \tag{4.5}$$

Let $U_k := X \setminus X_{d-k}$ and let $i_k : U_k \to U_{k+1}$ and $j_k : X_{d-k} \setminus X_{d-k-1} \to U_{k+1}$ be the usual inclusions.

Let us denote by $D^b_{cc}(X)$ the bounded derived category of cohomologically constructible sheaves of rational vector spaces on X with the previous stratification. Fix a perversity $\bar{p}$ and let us consider the following sets of properties in this category:

1. We say that $B^\bullet \in D^b_{cc}(X)$ verifies $[AX1]_k$ for perversity $\bar{p}$ if:

   (a) $B^\bullet_{|U_2}$ is quasi-isomorphic to $\mathbb{Q}_{U_2}$,

   (b) the cohomology sheaf $\mathcal{H}^i(B^\bullet)$ is 0 if $i \notin \{0, 1, ..., d\}$,

   (c_k) $\mathcal{H}^i(j_k^* B^\bullet_{|U_{k+1}})$ is equal to 0 if $i > \bar{p}(k)$,

   (d_k) the natural morphism $\mathcal{H}^i(j_k^* B^\bullet_{|U_{k+1}}) \to \mathcal{H}^i(j_k^* i_{k*} B^\bullet_{|U_k})$ is an isomorphism if $i \leq \bar{p}(k)$.

2. Let $\bar{q}$ be the complementary perversity of $\bar{p}$. We say that $B^\bullet \in D^b_{cc}(X)$ verifies $[AXS1]_k$ for perversity $\bar{p}$ if:

   (a) $B^\bullet_{|U_2}$ is quasi-isomorphic to $\mathbb{Q}_{U_2}$,

   (b) the cohomology sheaf $\mathcal{H}^i(B^\bullet)$ is 0 if $i \notin \{0, 1, ..., n\}$,

   (c_k) $\mathcal{H}^i(j_k^* B^\bullet_{|U_{k+1}})$ is equal to 0 if $i \leq \bar{q}(k)$,

   (d_k) the natural morphism $\mathcal{H}^i(j_k^* B^\bullet_{|U_{k+1}}) \to \mathcal{H}^i(j_k^* i_{k*} B^\bullet_{|U_k})$ is an isomorphism if $i > \bar{q}(k)$.

Remark 4.5.1. $B^\bullet$ verifies $[AX1]_k$ for $k = 2, ..., d$ if and only if $B^\bullet[d]$ verifies the axioms [AX1] of [GM83, section 3.3], that is, if $B^\bullet[d]$ is the intersection cohomology complex of X. So, we will denote an object of $D^b_{cc}(X)$ verifying $[AX1]_k$ for $k = 2, ..., d$ by $IC_{\bar{p}}[-d]$.

Definition 4.5.2. An *intersection space complex* of X with perversity $\bar{p}$ and stratification (4.5) is a complex of sheaves verifying $[AXS1]_k$ for $k = 2, ..., d$.

We denote by $IS_{\bar{p}}$ a complex of sheaves in X with these properties.

Remark 4.5.3. If the stratification of X induces a conical structure (see Definition 4.2.4) and there exist an intersection space pair of X with perversity $\bar{p}$ in the sense of Definition 4.2.22, then there exist an intersection space complex of X (see Theorem 4.4.18).

In the sequel, we will need equivalent axioms to $[AXS1]_k$. In the following remark, we review the method of [GM83, section 3.4] to get equivalent axioms to $[AX1]_k$ and $[AXS1]_k$.

Remark 4.5.4. Using the long exact sequence of cohomology associated to the distinguished triangle

$$j_k^! B^\bullet_{|U_{k+1}} \to j_k^* B^\bullet_{|U_{k+1}} \to j_k^* i_{k*} B^\bullet_{|U_k} \xrightarrow{[1]}$$

(c_k) and (d_k) of $[AX1]_k$ are equivalent to (c_k) and(d'_k) where (d'_k) is the following property:

$\mathcal{H}^i(j_k^! B^\bullet_{|U_{k+1}}) = 0$ if $i \le \bar{p}(k)+1$.

Given $x \in X_{d-k} \setminus X_{d-k-1}$, let $u_x : \{x\} \to X_{d-k} \setminus X_{d-k-1}$ and $j_x : \{x\} \to X$ be the canonical inclusions. Then, applying the property 1.13(15) of [GM83],

$$j_x^! B^\bullet = u_x^! j_k^! B^\bullet_{|U_{k+1}} = u_x^* j_k^! B^\bullet_{|U_{k+1}}[k-d]$$

So, (d'_k) is equivalent to the following property (d''_k):
For every $x \in X_{d-k} \setminus X_{d-k-1}$, $\mathcal{H}^i(j_x^! B^\bullet_{|U_{k+1}}) = 0$ if $i \le \bar{p}(k)+1+d-k = d-\bar{q}(k)-1$.

Now, we apply the same method to properties $[AXS1]_k$. Using again the long exact sequence of cohomology associated to

$$j_k^! B^\bullet_{|U_{k+1}} \to j_k^* B^\bullet_{|U_{k+1}} \to j_k^* i_{k*} B^\bullet_{|U_k} \xrightarrow{[1]},$$

we deduce (d_k) of $[AXS1]_k$ is equivalent to $(d1_k)$ and$(d2_k)$ where $(d1_k)$ and$(d2_k)$ are the following properties.

$(d1_k)$ $\mathcal{H}^i(j_k^! B^\bullet_{|U_{k+1}}) = 0$ if $i > \bar{q}(k)+1$.

$(d2_k)$ The canonical morphism $\mathcal{H}^{\bar{q}(k)+1}(j_k^! B^\bullet_{|U_{k+1}}) \to \mathcal{H}^{\bar{q}(k)+1}(j_k^* B^\bullet_{|U_{k+1}})$ is the morphism 0.

Moreover, using the property 1.13(15) of [GM83], these properties are equivalent to:

$(d1'_k)$ For every $x \in X_{d-k} \setminus X_{d-k-1}$, $\mathcal{H}^i(j_x^! B^\bullet_{|U_{k+1}}) = 0$ if $i > \bar{q}(k)+1+d-k = d-\bar{p}(k)-1$.

$(d2'_k)$ For every $x \in X_{d-k} \setminus X_{d-k-1}$, the canonical morphism

$$\mathcal{H}^{d-\bar{p}(k)-1}(j_x^! B^\bullet_{|U_{k+1}}) \to \mathcal{H}^{\bar{q}(k)+1}(j_x^* B^\bullet_{|U_{k+1}})$$

(given by property 1.13(15) of [GM83]) is the morphism 0.

Now, we recall useful definitions to compare the axioms $[AX1]_k$ with $[AXS1]_k$.

Definition 4.5.5. Let $B^\bullet$ be a complex of sheaves in a topological space X and, for every $x \in X$, let $j_x : \{x\} \to X$ be the canonical inclusion. Then,

- The *local support* of $B^\bullet$ in degree m is

$$\{x \in X | \mathcal{H}^m(j_x^* B^\bullet) \neq 0\}$$

- The *local cosupport* of $B^\bullet$ in degree m is

$$\{x \in X | \mathcal{H}^m(j_x^! B^\bullet) \neq 0\}$$

The properties (c_k) of $[AX1]_k$, (c_k) of $[AXS1]_k$ and (d'_k), $(d1'_k)$, $(d2'_k)$ of Remark 4.5.4 can be defined in terms of support and cosupport.

Let us consider a complex topological pseudomanifold X. Then, the upper middle perversity and the lower middle perversity (see Definition 2.3.2) are equal over the codimension of the strata of X.

Let $\bar{m}$ be the middle perversity. The following table, taken from [CaMi], illustrate the conditions of support and cosupport for a complex of sheaves $IC_{\bar{m}}[-d]$ verifying $[AX1]_k$ with perversity $\bar{m}$ for $k = 2, ..., d$.

degree					
8	c	c	c	c	c
7			c	c	c
6				c	c
5					c
4					
3					$\times$
2				$\times$	$\times$
1			$\times$	$\times$	$\times$
0	$\times$	$\times$	$\times$	$\times$	$\times$
	0	1	2	3	4

complex codimension
of the strata

The symbol c means the complex can have local cosupport at that place, while the symbol $\times$ means the complex can have local support at that place.

The following tables illustrate the conditions of support and cosupport for an intersection space complex $IS_{\bar{m}}$ with perversity $\bar{m}$.

degree					
8		$\times$	$\times$	$\times$	$\times$
7		$\times$	$\times$	$\times$	$\times$
6		$\times$	$\times$	$\times$	$\times$
5		$\times$	$\times$	$\times$	$\times$
4		$\times$	$\times$	$\times$	$\times^*$
3		$\times$	$\times$	$\times^*$	
2		$\times$	$\times^*$		
1		$\times^*$			
0	$\times$				
	0	1	2	3	4

complex codimension
of the strata

degree					
8	c				
7		c^*			
6		c	c^*		
5		c	c	c^*	
4		c	c	c	c^*
3		c	c	c	c
2		c	c	c	c
1		c	c	c	c
0		c	c	c	c
	0	1	2	3	4

complex codimension
of the strata

The symbol c means the complex can have local cosupport at that place, while the symbol $\times$ means the complex can have local support at that place. Moreover, the symbol $*$ means the support and the cosupport must verify a special condition given by $(d2'_k)$.

Note that in U_2, $(IS_{\bar{m}})_{|U_2} \cong (IC_{\bar{m}}[-d])_{|U_2} \cong \mathbb{Q}_{U_2}$. However, in X_{d-2}, the place at which $IS_{\bar{m}}$ can have support is exactly the place at which $IC_{\bar{m}}[-d]$ cannot have support and the place at which $IS_{\bar{m}}$ can have cosupport is exactly the place at which $IC_{\bar{m}}[-d]$ cannot have cosupport.

4.6 A constructible complex approach to intersection space complexes

In this section, we study necessary and sufficient conditions for the existence of an intersection space complex of X with a perversity $\bar{p}$. Unlike intersection cohomology complexes, intersection space complexes are not unique. We study the space parametrizing the different choices of intersection space complexes for a fixed perversity.

4.6.1 Homological algebra review

We need the following lemma, that should be well known, we include a proof for convenience of the reader.

Lemma 4.6.1. *Let*

$$\begin{tikzcd} A^\bullet \arrow[r, "f"] & B^\bullet \arrow[r, "g"] & C^\bullet \arrow[ll, bend left, "\phi"', "{[1]}"] \end{tikzcd}$$

be a distinguished triangle in the category $D^b_{cc}(X)$.

The following four conditions are equivalent:

(1) f admits a retract

(2) g admits a section

(3) There is an isomorphism in the derived category $\gamma : B^\bullet \cong A^\bullet \oplus C^\bullet$ such that $f = \gamma^{-1} \circ i_A$ and $g = p_C \circ \gamma$ where $i_A : A^\bullet \to A^\bullet \oplus C^\bullet$ is the natural inclusion and $p_C : A^\bullet \oplus C^\bullet \to C^\bullet$ is the natural projection.

(4) ϕ is the morphism 0

Proof. First we prove that (1) implies (3). Let r be a retract of f. Then, $r \circ f = Id_{A^\bullet}$. So, the morphism induced by f between the cohomology sheaves $\mathcal{H}^i(f) : \mathcal{H}^i(A^\bullet) \to \mathcal{H}^i(B^\bullet)$ is injective for every $i \in \mathbb{Z}$. Hence, using the long exact sequence of cohomology

$$... \to \mathcal{H}^{i-1}(C^\bullet) \xrightarrow{\mathcal{H}^{i-1}(\phi)} \mathcal{H}^i(A^\bullet) \xrightarrow{\mathcal{H}^i(f)} \mathcal{H}^i(B^\bullet) \xrightarrow{\mathcal{H}^i(g)} \mathcal{H}^i(C^\bullet) \xrightarrow{\mathcal{H}^i(\phi)} \mathcal{H}^{i+1}(A^\bullet) \to ...,$$

we deduce $\mathcal{H}^i(\phi)=0$ for every $i\in\mathbb{Z}$. Consequently, we have short exact sequences

$$0\to\mathcal{H}^i(A^\bullet)\xrightarrow{\mathcal{H}^i(f)}\mathcal{H}^i(B^\bullet)\xrightarrow{\mathcal{H}^i(g)}\mathcal{H}^i(C^\bullet)\to 0 \tag{4.6}$$

Now, let us consider the morphism

$$\gamma=\begin{pmatrix} r\\ g\end{pmatrix}:B^\bullet\to A^\bullet\oplus C^\bullet$$

The morphism induced by γ between the cohomology sheaves is

$$\mathcal{H}^i(\gamma)=\begin{pmatrix} \mathcal{H}^i(r)\\ \mathcal{H}^i(g)\end{pmatrix}:\mathcal{H}^i(B^\bullet)\to\mathcal{H}^i(A^\bullet)\oplus\mathcal{H}^i(C^\bullet),$$

which is an isomorphism, since $\mathcal{H}^i(r)$ is a retract of $\mathcal{H}^i(f)$ and (4.6) is exact.

Moreover, since $f\circ g=0$, we have $\gamma\circ f=i_A$ and it is clear that $p_C\circ\gamma=g$. So, we have proven (1) implies (3).

Now, we prove (2) implies (3). Let s be a section of g and let us consider the morphism

$$\gamma'=(f,s):A^\bullet\oplus C^\bullet\to B^\bullet$$

In the same way that in the previous implication, we can show that γ' is a quasi-isomorphism, and that we have $f=\gamma'\circ i_A$ and $g\circ\gamma'=p_C$. So, $\gamma=(\gamma')^{-1}$ is the isomorphism which appears in condition (3).

Moreover, if condition (3) is true, $p_A:A^\bullet\oplus C^\bullet\to A^\bullet$ denotes the natural projection and $i_C:C^\bullet\to A^\bullet\oplus C^\bullet$ denotes the natural inclusion, then $p_A\circ\gamma$ is a retract of f and $\gamma^{-1}\circ i_C$ is a section of g. So, (3) implies (1) and (2).

Now, it is enough to prove (3) $\Leftrightarrow$ (4). (3) implies that we have the following isomorphism between distinguished triangles:

$$\begin{tikzcd}
A^\bullet \arrow[r, "f"] \arrow[d, "Id_{A^\bullet}"'] & B^\bullet \arrow[r, "g"] \arrow[d, "\gamma"] & C^\bullet \arrow[d, "Id_{C^\bullet}"] \arrow[ll, bend right, "\overset{[1]}{\phi}"'] \\
A^\bullet \arrow[r, "i_A"] & A^\bullet\oplus C^\bullet \arrow[r, "p_C"] & C^\bullet \arrow[ll, bend left, "\underset{[1]}{0}"]
\end{tikzcd}$$

So, ϕ is the morphism 0.

Now, suppose $\phi=0$ and let us prove condition (3). By properties of the triangulated categories, we know

$$\begin{tikzcd}
C^\bullet[-1] \arrow[r, "-\phi[-1]"] & A^\bullet \arrow[r, "f"] & B^\bullet \arrow[ll, bend left, "\underset{[1]}{g}"]
\end{tikzcd}$$

is a distinguished triangle. So, if $\phi = 0$, there is an isomorphism $\gamma : B^\bullet \cong A^\bullet \oplus C^\bullet$ which completes the following isomorphism of triangles

$$\begin{tikzcd}
C^\bullet[-1] \arrow[r, "0"] \arrow[d, "Id_{C^\bullet}[-1]"'] & A^\bullet \arrow[r, "f"] \arrow[d, "Id_{A^\bullet}"] & B^\bullet \arrow[d, "\gamma"] \arrow[ll, bend right, "g"', "{[1]}"] \\
C^\bullet[-1] \arrow[r, "0"] & A^\bullet \arrow[r, "i_A"] & A^\bullet \oplus C^\bullet \arrow[ll, bend left, "p_C", "{[1]}"']
\end{tikzcd}$$

γ is the isomorphism which appears in condition (3).

□

Definition 4.6.2. The triangle

$$\begin{tikzcd}
A^\bullet \arrow[r, "f"] & B^\bullet \arrow[r, "g"] & C^\bullet \arrow[ll, bend left, "\phi"', "{[1]}"]
\end{tikzcd}$$

is said to be split if it verifies the conditions of Lemma 4.6.1.

4.6.2 Charaterization of existence and study of uniqueness

Now, we can state the main theorem in this section.

Theorem 4.6.3. *The following holds:*

1. *[Goreski-McPherson] There exist one object in $D^b_{cc}(X)$ verifying $[AX1]_k$ for $k = 2, ..., d$. This object is unique up to isomorphism.*

2. *Suppose there exist an intersection space complex in U_r, IS_{r-1}, that is, IS_{r-1} is an object in $D^b_{cc}(U_r)$ and it verifies $[AXS1]_k$ for $k = 2, ..., r-1$. Then, there exist an intersection space complex in U_{r+1}, IS_r, such that $(IS_r)_{|U_r} \cong IS_{r-1}$ if and only if the distinguised triangle:*

$$\tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1} \xrightarrow{f} j_{r*} j_r^* i_{r*} IS_{r-1} \to \tau_{> \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1} \xrightarrow{[1]} \qquad (4.7)$$

is split. Moreover, there is a bijection

$$\left\{ \begin{array}{c} \text{intersection space complexes} \\ IS_r \in D^b_{cc}(U_{r+1}) \text{ such} \\ \text{that} \, (IS_r)_{|U_r} \cong IS_{r-1} \end{array} \right\} / \{\text{isomorphism}\} \longleftrightarrow \{\text{retracts of} \, f\} / \sim$$

where $\sim$ is the equivalence relation such that $\lambda_1 \sim \lambda_2$ if and only if there exit isomorphisms $\alpha : \tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1} \to \tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}$ and $\beta : i_{r*} IS_{r-1} \to i_{r*} IS_{r-1}$ such that $\lambda_2 = \alpha \circ \lambda_1 \circ j_{r*} j_r^* \beta$.

Proof. **1.** is proved in [GM83, section 3], but we give proof adapted to our needs.

There exist a unique object (up to isomorphism) $\mathbb{Q}_{U_2}$ in $D^b_{cc}(U_2)$ verifying (a) and (b) of $[AX1]_k$. Suppose there exist a unique object (up to isomorphism) $IC_{\bar{p}\,r-1}[-d]$ in $D^b_{cc}(U_r)$ verifying $[AX1]_k$ for $k = 2, ..., r-1$, and consider the following composition of natural morphisms:

$$\begin{tikzcd} i_{r*}IC_{\bar{p}\,r-1}[-d] \arrow[r] \arrow[rr, bend left, "\phi_r"] & j_{r*}j_r^*i_{r*}IC_{\bar{p}\,r-1}[-d] \arrow[r] & \tau_{>\bar{p}(r)}j_{r*}j_r^*i_{r*}IC_{\bar{p}\,r-1}[-d] \end{tikzcd}$$

Let us define $IC_{\bar{p}\,r}[-d] := cone(\phi_r)[-1]$. Then, there is a distinguished triangle:

$$IC_{\bar{p}\,r}[-d] \to i_{r*}IC_{\bar{p}\,r-1}[-d] \xrightarrow{\phi_r} \tau_{>\bar{p}(r)}j_{r*}j_r^*i_{r*}IC_{\bar{p}\,r-1}[-d] \xrightarrow{[1]} \tag{4.8}$$

Using the long exact sequence of cohomology associated to this triangle we can prove that $IC_r[-d]$ verifies $[AX1]_k$ for $k = 2, ..., r$.

Now, suppose there exist another object $B^\bullet$ in $D^b_{cc}(U_{r+1})$ verifying $[AX1]_k$ for $k = 2, ..., r$. Then, $B^\bullet_{|U_r}$ verifies $[AX1]_k$ for $k = 2, ..., r-1$. So, there exist an isomorphism $B^\bullet_{|U_r} \cong IC_{\bar{p}\,r-1}[-d]$.

Let $\varphi : B^\bullet \to i_{r*}IC_{\bar{p}\,r-1}[-d]$ be the composition of the canonical morphism $B^\bullet \to i_{r*}B^\bullet_{U_r}$ and an isomorphism $i_{r*}B^\bullet_{|U_r} \cong i_{r*}IC_{\bar{p}\,r-1}[-d]$ and let $C^\bullet := cone(\varphi)$. Since $i^*\varphi$ is an isomorphism, we have the isomorphism $i^*C^\bullet \cong 0$ in the derived category.

Then, the distinguised triangle

$$i_{r!}i_r^*C^\bullet \to C^\bullet \to j_{r*}j_r^*C^\bullet \xrightarrow{[1]}$$

implies there exist an isomorphism $C^\bullet \cong j_{r*}j_r^*C^\bullet$.

Moreover, with the long exact sequence of cohomology associated to

$$j_{r*}j_r^*B^\bullet \xrightarrow{j_{r*}j_r^*\varphi} j_{r*}j_r^*i_{r*}IC_{\bar{p}\,r-1}[-n] \xrightarrow{\psi} j_{r*}j_r^*C^\bullet$$

we prove $\mathcal{H}^i(j_{r*}j_r^*C^\bullet) = 0$ if $i \leq \bar{p}(r)$ and $\mathcal{H}^i(\psi) : \mathcal{H}^i(j_{r*}j_r^*i_{r*}IC_{\bar{p}\,r-1}[-d]) \to \mathcal{H}^i(j_{r*}j_r^*C^\bullet)$ is an isomorphism if $i > \bar{p}(r)$. Then, we obtain isomorphisms

$$C^\bullet \cong \tau_{>\bar{p}(r)}j_{r*}j_r^*i_{r*}IC_{\bar{p}\,r-1}[-d] \qquad \text{and} \qquad B^\bullet \cong IC_{\bar{p}\,r}[-d].$$

Repeating this process finitely we obtain $IC_{\bar{p}}[-d] \in D^b_{cc}(X)$ verifying $[AX1]_k$ for $k = 2, ..., d$ and it is unique up to isomorphism.

Now, we prove **2.** Let IS_{r-1} be an intersection space complex in U_r. We have to prove that there is a bijective map

$$\left\{ \begin{array}{c} \text{intersection space complexes} \\ IS_r \in D^b_{cc}(U_{r+1}) \text{ such} \\ \text{that } (IS_r)_{|U_r} \cong IS_{r-1} \end{array} \right\} / \{\text{isomorphism}\} \longleftrightarrow \{\text{retracts of } f\} / \sim$$

Let λ be a retract of f and consider the following composition of morphisms:

$$\begin{tikzcd} i_{r*}IS_{r-1} \arrow[r, "a"] \arrow[rr, bend left, "\varphi_\lambda"] & j_{r*}j_r^*i_{r*}IS_{r-1} \arrow[r, "\lambda"] & \tau_{\leq \bar{q}(r)}j_{r*}j_r^*i_{r*}IS_{r-1} \end{tikzcd}$$

where a is the canonical morphism.

Let us define $IS_r := cone(\varphi_\lambda)[-1]$. Then, there is a distinguised triangle:

$$IS_r \to i_{r*}IS_{r-1} \xrightarrow{\varphi_\lambda} \tau_{\leq \bar{q}(r)}j_{r*}j_r^*i_{r*}IS_{r-1} \xrightarrow{[1]} \tag{4.9}$$

Since $(j_{r*}j_r^*i_{r*}IS_{r-1})_{|U_r}$ equals 0, $(IS_r)_{|U_r}$ is isomorphic to IS_r and, using the long exact sequence associated to the triangle, one proves that IS_r verifies $[AXS1]_k$ for $k = 2, ..., r$.

Let λ' be a different retract of f such that $\lambda \sim \lambda'$. We have to prove $cone(\varphi_\lambda) \cong cone(\varphi_{\lambda'})$.

Let $\alpha : \tau_{\leq \bar{q}(r)}j_{r*}j_r^*i_{r*}IS_{r-1} \to \tau_{\leq \bar{q}(r)}j_{r*}j_r^*i_{r*}IS_{r-1}$ and $\beta : i_{r*}IS_{r-1} \to i_{r*}IS_{r-1}$ be isomorphisms such that $\lambda' = \alpha \circ \lambda \circ j_{r*}j_r^*\beta$.

Since we know the equalities $\varphi_\lambda = \lambda \circ a$ and $\varphi_{\lambda'} = \lambda' \circ a$, we have to prove the isomorphism $cone(\lambda \circ a) \cong cone(\alpha \circ \lambda \circ j_{r*}j_r^*\beta \circ a)$. Moreover, since α is an isomorphism, $cone(\alpha \circ \lambda \circ j_{r*}j_r^*\beta \circ a)$ is isomorphic to $cone(\lambda \circ j_{r*}j_r^*\beta \circ a)$. Now, consider the following diagrams associated to the octahedral axiom of distinguished triangles.

$$\begin{tikzcd} i_{r*}IS_{r-1} \arrow[rr, "a"] \arrow[rrrr, bend left, "\lambda\circ a"] & & j_{r*}j_r^*i_{r*}IS_{r-1} \arrow[rr, "\lambda"] \arrow[dl] & & \tau_{\leq \bar{q}(r)}j_{r*}j_r^*i_{r*}IS_{r-1} \arrow[dl] \arrow[ddll, bend left] \\ & cone(a) \arrow[ul, "{[1]}"] \arrow[dr] & & cone(\lambda) \arrow[ul, "{[1]}"] \arrow[ll, "{[1]}"] & \\ & & cone(\lambda \circ a) \arrow[ur] \arrow[uull, bend left, "{[1]}"] & & \end{tikzcd} \tag{4.10}$$

$$\begin{tikzcd} i_{r*}IS_{r-1} \arrow[rr, "a"] \arrow[rrrr, bend left, "\lambda\circ j_{r*}j_r^*\beta\circ a"] & & j_{r*}j_r^*i_{r*}IS_{r-1} \arrow[rr, "\lambda\circ j_{r*}j_r^*\beta"] \arrow[dl] & & \tau_{\leq \bar{q}(r)}j_{r*}j_r^*i_{r*}IS_{r-1} \arrow[dl] \arrow[ddll, bend left] \\ & cone(a) \arrow[ul, "{[1]}"] \arrow[dr] & & cone(\lambda \circ j_{r*}j_r^*\beta) \arrow[ul, "{[1]}"] \arrow[ll, "{[1]}"] & \\ & & cone(\lambda \circ j_{r*}j_r^*\beta \circ a) \arrow[ur] \arrow[uull, bend left, "{[1]}"] & & \end{tikzcd} \tag{4.11}$$

Let $\phi : cone(\lambda \circ j_{r*}j_r^*\beta) \to cone(\lambda)$ be an isomorphism completing the following isomorphism between triangles

$$\begin{tikzcd}
j_{r*}j_r^*i_{r*}IS_{r-1} \arrow[r, "\lambda\circ j_{r*}j_r^*\beta"] \arrow[d, "j_{r*}j_r^*\beta"] & \tau_{\leq \bar{q}(r)}j_{r*}j_r^*i_{r*}IS_{r-1} \arrow[r] \arrow[d, "Id"] & cone(\lambda \circ j_{r*}j_r^*\beta) \arrow[d, "\phi"] \\
j_{r*}j_r^*i_{r*}IS_{r-1} \arrow[r, "\lambda"] & \tau_{\leq \bar{q}(r)}j_{r*}j_r^*i_{r*}IS_{r-1} \arrow[r] & cone(\lambda)
\end{tikzcd}$$

Then, if $\rho : cone(a) \to cone(a)$ is an isomorphism completing the triangles isomorphism

$$\begin{tikzcd}
i_{r*}IS_{r-1} \arrow[r, "a"] \arrow[d, "\beta"] & j_{r*}j_r^*i_{r*}IS_{r-1} \arrow[r] \arrow[d, "j_{r*}j_r^*\beta"] & cone(a) \arrow[d, "\rho"] \\
i_{r*}IS_{r-1} \arrow[r, "a"] & j_{r*}j_r^*i_{r*}IS_{r-1} \arrow[r] & cone(a)
\end{tikzcd}$$

the diagram

$$\begin{tikzcd}
cone(\lambda \circ j_{r*}j_r^*\beta) \arrow[r, "{[1]}"] \arrow[d, "\phi"] & cone(a) \arrow[d, "\rho"] \\
cone(\lambda) \arrow[r, "{[1]}"] & cone(a)
\end{tikzcd}$$

is commutative.

Therefore, there exist an isomorphism $cone(\lambda\circ j_{r*}j_r^*\beta\circ a) \cong cone(\lambda\circ a)$, completing the following morphism between triangles

$$\begin{tikzcd}
cone(\lambda \circ j_{r*}j_r^*\beta) \arrow[r, "{[1]}"] \arrow[d, "\phi"] & cone(a) \arrow[r] \arrow[d, "\rho"] & cone(\lambda \circ j_{r*}j_r^*\beta \circ a) \arrow[d] \\
one(\lambda) \arrow[r, "{[1]}"] & cone(a) \arrow[r] & cone(\lambda \circ a)
\end{tikzcd}$$

Now, suppose there exist an intersection space complex in U_{r+1}, IS_r such that $(IS_r)_{|U_r} \cong IS_{r-1}$. We have to prove that the triangle (4.7) is split.

Let $h : IS_r \to i_{r*}IS_{r-1}$ be the composition of the canonical morphism $IS_r \to i_{r*}(IS_r)_{|U_r}$ and an isomorphism $i_{r*}(IS_r)_{|U_r} \cong i_{r*}IS_{r-1}$ and let $C^\bullet := cone(h)$. Then, there is a distinguished triangle:

$$IS_r \xrightarrow{h} i_{r*}IS_{r-1} \xrightarrow{g} C^\bullet \xrightarrow{[1]}$$

Note that $i_r^*h : i_r^*IS_r \to IS_{r-1}$ is an isomorphism. So, $i_r^*C^\bullet$ is isomorphic to 0 in the derived category. Then, the canonical triangle

$$i_{r!}i_r^*C^\bullet \to C^\bullet \to j_{r*}j_r^*C^\bullet \xrightarrow{[1]}$$

implies the isomorphism $C^\bullet \cong j_{r*}j_r^*C^\bullet$.

Moreover, the long exact sequence of cohomology associated to the triangle

$$j_{r*}j_r^*IS_r \xrightarrow{j_{r*}j_r^*h} j_{r*}j_r^*i_{r*}IS_{r-1} \xrightarrow{j_{r*}j_r^*g} j_{r*}j_r^*C^\bullet \xrightarrow{[1]}$$

implies that

$$\mathcal{H}^i(j_{r*}j_r^*C^\bullet) \cong \begin{cases} 0 & \text{if}\, i > \bar{q}(r) \\ \mathcal{H}^i(j_{r*}j_r^*i_{r*}IS_{r-1}) & \text{if}\, i \le \bar{q}(r) \end{cases}$$

Applying the functor $\tau_{\le \bar{q}(r)}$ to $j_{r*}j_r^*g$, we obtain the following commutative diagram:

$$\begin{tikzcd} \tau_{\le\bar{q}(r)}j_{r*}j_r^*i_{r*}IS_{r-1} \arrow[r, "a"] \arrow[d, "f"] & \tau_{\le\bar{q}(r)}j_{r*}j_r^*C^\bullet \arrow[d, "c"] \\ j_{r*}j_r^*i_{r*}IS_{r-1} \arrow[r, "b"] & j_{r*}j_r^*C^\bullet \end{tikzcd} \tag{4.12}$$

where f and c are the canonical morphisms, $a = \tau_{\le\bar{q}(r)}j_{r*}j_r^*g$ and $b = j_{r*}j_r^*g$. Moreover, a and c are isomorphisms and the composition $\lambda = a^{-1} \circ c^{-1} \circ b$ is a retract of f. So, the triangle (4.7) is split by Lemma 4.6.1.

Let $IS'_r \in D^b_{cc}(U_{r+1})$ be isomorphic to IS_r. Then, we have an isomorphism $(IS'_r)_{|U_r} \cong IS_{r-1}$ and IS'_r is an intersection space complex in U_{r+1}.

Let $h' : IS'_r \to i_{r*}IS_{r-1}$ be the composition of the canonical restriction morphism and an isomorphism $i_{r*}(IS'_r)_{|U_r} \cong i_{r*}IS_{r-1}$, let $K^\bullet := \mathit{cone}(h')$ and consider the triangle

$$IS'_r \xrightarrow{h'} i_{r*}IS_r \xrightarrow{g'} K^\bullet \xrightarrow{[1]}$$

Applying the functor $\tau_{\le\bar{q}(r)}$ to $j_{r*}j_r^*g'$, we obtain the following commutative diagram:

$$\begin{tikzcd} \tau_{\le\bar{q}(r)}j_{r*}j_r^*i_{r*}IS_{r-1} \arrow[r, "a'"] \arrow[d, "h"] & \tau_{\le\bar{q}(r)}j_{r*}j_r^*K^\bullet \arrow[d, "c'"] \\ j_{r*}j_r^*i_{r*}IS_{r-1} \arrow[r, "b'"] & j_{r*}j_r^*K^\bullet \end{tikzcd} \tag{4.13}$$

where h and c' are the canonical morphisms, $a' = \tau_{\le\bar{q}(r)}j_{r*}j_r^*g'$ and $b' = j_{r*}j_r^*g'$. Moreover, a' and c' are isomorphisms and $\lambda' := a'^{-1} \circ c'^{-1} \circ b'$ is a retract of f.

Let $\alpha : IS_r \to IS'_r$ be an isomorphism; let h be the composition of the canonical morphism $IS_r \to i_{r*}(IS_r)_{|U_r}$ and an isomorphism $\gamma : i_{r*}(IS_r)_{|U_r} \to i_{r*}IS_{r-1}$; let h' be the composition of the canonical morphism $IS'_r \to i_{r*}(IS'_r)_{|U_k}$ and an isomorphism $\gamma' : i_{r*}(IS'_r)_{|U_k} \to i_{r*}IS'_{r-1}$. Finally define $\beta := \gamma' \circ i_{r*}i_r^*\alpha \circ \gamma^{-1}$. Then, there is an isomorphism between triangles:

$$\begin{tikzcd} j_{r*}j_r^*IS_r \arrow[r, "j_{r*}j_r^*h"] \arrow[d, "j_{r*}j_r^*\alpha"] & j_{r*}j_r^*i_{r*}IS_{r-1} \arrow[r, "j_{r*}j_r^*g"] \arrow[d, "j_{r*}j_r^*\beta"] & j_{r*}j_r^*C^\bullet \arrow[d, "\delta"] \\ j_{r*}j_r^*IS'_r \arrow[r, "j_{r*}j_r^*h'"] & j_{r*}j_r^*i_{r*}IS_{r-1} \arrow[r, "j_{r*}j_r^*g'"] & j_{r*}j_r^*K^\bullet \end{tikzcd}$$

where α, β and δ are isomorphisms.

Then, the morphisms of diagrams (4.12) and (4.13) have the following relations:

$$a' = \tau_{\leq \bar{q}(r)} \delta \circ a \circ (\tau_{\leq \bar{q}(r)} j_{r*} j_r^* \beta)^{-1}$$

$$b' = \delta \circ a \circ (j_{r*} j_r^* \beta)^{-1}$$

$$c' = \delta \circ c \circ (\tau_{\leq \bar{q}(r)} \delta)^{-1}$$

and, we obtain $\lambda = (\tau_{\leq \bar{q}(r)} j_{r*} j_r^* \beta)^{-1} \circ \lambda' \circ j_{r*} j_r^* \beta$. So, λ is equivalent to λ' by the equivalence relation $\sim$. □

Remark 4.6.4. There is a unique object in $D^b_{cc}(U_2)$ up to isomorphism verifying (a) and (b) from $[AXS1]_k$, $\mathbb{Q}_{U_2}$.

Now we are going to study the equivalence relation $\sim$, which appears in Theorem 4.6.3.

Remember that two retracts λ_1 and λ_2 of $\tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1} \xrightarrow{f} j_{r*} j_r^* i_{r*} IS_{r-1}$ are equivalent by $\sim$ if and only if there exit isomorphisms

$$\alpha : \tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1} \to \tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}$$

$$\beta : i_{r*} IS_{r-1} \to i_{r*} IS_{r-1}$$

such that $\lambda_2 = \alpha \circ \lambda_1 \circ j_{r*} j_r^* \beta$.

Let $\alpha : \tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1} \to \tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}$ and $\beta : i_{r*} IS_{r-1} \to i_{r*} IS_{r-1}$ be isomorphisms and let λ be a retract of f. Then, $\alpha \circ \lambda \circ j_{r*} j_r^* \beta$ is a retract of f if and only if $\alpha \circ \lambda \circ j_{r*} j_r^* \beta \circ f = Id$, that is, if α is equal to $(\lambda \circ j_{r*} j_r^* \beta \circ f)^{-1}$. So, α is determined by β and λ and the set of retracts of f which are equivalent to λ is determined by the automorphisms $i_{r*} IS_{r-1} \to i_{r*} IS_{r-1}$.

Since i_r^* is left-adjoint to i_{r*}, the space of automorphisms $Aut(i_{r*} IS_{r-1})$ is isomorphic to the space of isomorphisms $Iso(i_r^* i_{r*} IS_{r-1}, IS_{r-1}) = Aut(IS_{r-1})$.

4.6.3 The space of obstructions to existence and uniqueness.

In particular, if r is the dimension of the largest non-trivial stratum, we have $U_r = U_2$ and $Aut(IS_{r-1})$ is isomorphic to $Aut(\mathbb{Q}_{U_2})$, which is the space of homothetic transformations. Moreover, if $\beta \in Aut(i_{r*} IS_{r-1})$ is a homothetic transformation $\alpha = (\lambda \circ j_{r*} j_r^* \beta \circ f)^{-1}$ is the inverse homothetic transformation. So,if r is the dimension of the largest non-trivial stratum, the equivalence relation $\sim$ is trivial. Consequently, there is a bijective map

$$\left\{ \begin{array}{c} \text{intersection space complexes} \\ IS_r \in D^b_{cc}(U_{r+1}) \end{array} \right\} / \{\text{isomorphism}\} \longleftrightarrow \{\text{retracts of}\, f\}$$

Remark 4.6.5. For any r, the triangle (4.7) induces a exact sequence

$$... \to [\tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}, \tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}[-1]] \to$$

$$\to [\tau_{>\bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}, \tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}] \xrightarrow{\widetilde{g}} [j_{r*} j_r^* i_{r*} IS_{r-1}, \tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}] \xrightarrow{\widetilde{f}}$$

$$\xrightarrow{\widetilde{f}} [\tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}, \tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}] \to ...$$

Moreover, the retracts of f are de elements $\lambda \in [j_{r*} j_r^* i_{r*} IS_{r-1}, \tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}]$ such that $\widetilde{f}(\lambda)$ is the identity. So, the space {retracts of f} is modulated by the vector space

$$[\tau_{>\bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}, \tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}]$$

The following is an inmediate consequence of the previous construction.

Corollary 4.6.6. *Suppose there exist an intersection space complex IS_{r-1} in U_r, that is, IS_{r-1} is an object in $D^b_{cc}(U_r)$ and it verifies $[AXS1]_k$ for $k = 2, ..., r-1$.*

- *The obstruction to existence of intersection space in the next stratum lives in*

$$Ext^1(\tau_{>\bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}, \tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}) =$$

$$= [\tau_{>\bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}, \tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}[1]].$$

- *The obstructions for uniqueness live in the group*

$$Hom(\tau_{>\bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}, \tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}) =$$

$$= [\tau_{>\bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}, \tau_{\leq \bar{q}(r)} j_{r*} j_r^* i_{r*} IS_{r-1}],$$

  modulo the equivalence relation described above. The equivalence relation is trivial for the second stratum.

4.7 Classes of spaces admiting Intersection Space Complexes and counterexamples

In this section we provide some examples and counterexamples to illustrate our theory.

First, we introduce two classes of varieties which admit an intersection space complex for every perversity. The first class depends on the tubular neighbourhoods of the strata: if every stratum admits a trivial tubular neighbourhood, then there exist the intersection space complex for every perversity. The second class depends on the dimension of the strata: if every singular stratum has homological dimension for locally constant sheaves bounded by 1, then there exist the intersection space complex for every perversity.

Let X be a topological pseudomanifold with the following stratification:

$$X = X_d \supset X_{d-2} \supset ... \supset X_0 \supset X_{-1} = \emptyset$$

For $k = 2, ..., d+1$, let $U_k := X \setminus X_{d-k}$. We will also denote by $i_k : U_k \to U_{k+1}$, $j_k : X_{d-k} \setminus X_{d-k-1} \to U_{k+1}$ and $i_{k_1,k_2} : U_{k_1} \to U_{k_2}$ the canonical inclusions.

Definition 4.7.1. If $B^\bullet_{k-1}$ is a complex of sheaves in U_k, we say $B^\bullet_{k-1}$ verifies the property (P_r), where $r \geq k$, if $j_r^* i_{k,r+1*} B^\bullet_{k-1}$ is quasi-isomorphic to a complex of constant sheaves.

Remark 4.7.2. If $B^\bullet_{k-1}$ is a complex of sheaves in U_k quasi-isomorphic to a complex of constant sheaves and (X, X_{d-2}) has a conical structure which verifies the property (T_r) of Definition 4.2.8 for some $r \geq k$, then $B^\bullet_{k-1}$ verifies the property (P_r).

Proposition 4.7.3. *Given $k \in \{2, ..., d\}$, if there exist an intersection space with perversity $\bar{p}$ in U_k, IS_{k-1}, which verifies (P_k), then there exist an intersection space with perversity $\bar{p}$ in U_{k+1}, IS_k, such that $(IS_k)_{|U_k}$ is quasi-isomorphic to IS_{k-1}.*

Proof. $j_k^* i_{k*} IS_{k-1}$ is (up to isomorphism in the derived category) a complex of constant sheaves. So, the triangle

$$\tau_{\leq q} j_{k*} j_k^* i_{k*} IS_{k-1} \to j_{k*} j_k^* i_{k*} IS_{k-1} \to \tau_{>q} j_{k*} j_k^* i_{k*} IS_{k-1} \xrightarrow{[1]}$$

is split for every $q \in \mathbb{Z}$ and, applying Theorem 4.6.3, we conclude. □

Lemma 4.7.4. *Let us suppose that there exist an intersection space complex with perversity $\bar{p}$ in U_k, IS_{k-1}. If $(IS_{k-1})_{|U_{k-1}}$ verifies the properties (P_{k-1}) and (P_r) for some $r \geq k$ and (X, X_{d-2}) has a conical structure which verifies the property (T_r) of Definition 4.2.8, then IS_{k-1} verifies the property (P_r).*

Proof. By Theorem 4.6.3, $j_r^* i_{k,r+1*} IS_{k-1}[1]$ is quasi-isomorphic to the cone of a morphism

$$j_r^* i_{k-1,r+1*} (IS_{k-1})_{|U_{k-1}} \to j_r^* i_{k,r+1*} \tau_{\leq \bar{q}(k-1)} j_{k-1*} j_{k-1}^* i_{k-1*} (IS_{k-1})_{|U_{k-1}}$$

Since $(IS_{k-1})_{|U_{k-1}}$ verifies the properties (P_{k-1}) and (P_r), $j_r^* i_{k-1,r+1*} (IS_{k-1})_{|U_{k-1}}$ and $\tau_{\leq \bar{q}(k-1)} j_{k-1}^* i_{k-1*} (IS_{k-1})_{|U_{k-1}}$ are quasi-isomorphic to complexes of constant sheaves. Then, using that (X, X_{d-2}) has a conical structure verifying the property (T_r), we can prove that $j_r^* i_{k,r+1*} \tau_{\leq \bar{q}(k-1)} j_{k-1*} j_{k-1}^* i_{k-1*} (IS_{k-1})_{|U_{k-1}}$ is also quasi-isomorphic to a complex of constant sheaves. So, $j_r^* i_{k,r+1*} IS_{k-1}$ is quasi isomorphic to a complex of constant sheaves and we conclude. □

Corollary 4.7.5. *If the pair (X, X_{d-2}) has a conical structure which verifies the property (T_t) of Definition 4.2.8 for any t, then there exist the intersection space complex of X for every perversity.*

Proof. The constant sheaf $\mathbb{Q}_{U_2}$ verifies (P_r) for every $r \geq 2$ such that the pair (X, X_{d-2}) has a conical structure with the property (T_r). So, if (X, X_{d-2}) has a conical structure which verifies the property (T_r) for any r, using Lemma 4.7.4 and Proposition 4.7.3, we can construct inductively for every k an intersection space complex with perversity $\bar{p}$ in U_k which verifies (P_r) for every $r \geq k$. □

Example 4.7.6. Since toric varieties verify the hypothesis of Corollary 4.7.5, every toric variety admits an intersection space complex for every perversity.

Definition 4.7.7. A space Y has homological dimension for locally constant sheaves bounded by m if any locally constant sheaf in Y has no cohomology in degree higher than m.

Theorem 4.7.8. *Let X be a topological pseudomanifold with the following stratification:*

$$X = X_d \supset X_{d-2} \supset ... \supset X_0 \supset X_{-1} = \emptyset$$

such that X_{d-2} has homological dimension for locally constant sheaves bounded by 1.

Then, there exist the intersection space complex of X for every perversity $\bar{p}$.

Moreover, if X_{d-2} has homological dimension for locally constant sheaves bounded by 0, the intersection space complex is unique.

Proof. To prove the existence it is enough to prove that, for any topological space Y which has homological dimension for locally constant sheaves bounded by 1, any complex of sheaves $B^\bullet$ in Y and any integer m, the triangle

$$\tau_{\leq m}B^\bullet \to B^\bullet \to \tau_{>m}B^\bullet \xrightarrow{\phi} \tau_{\leq m}B^\bullet[1] \to ...$$

is split.

This triangle is split if and only the morphism

$$\phi \in Ext^1(\tau_{>m}B^\bullet, \tau_{\leq m}B^\bullet)$$

is 0.

$Ext^1(\tau_{>m}B^\bullet, \tau_{\leq m}B^\bullet)$ is the first hipercohomology group of the complex of sheaves $\mathcal{H}om^\bullet(\tau_{>m}B^\bullet, \tau_{\leq m}B^\bullet)$.

Let $E_r^{p,q}$ be the local to global spectral sequence of $\mathcal{H}om^\bullet(\tau_{\leq m}B^\bullet, \tau_{>m}B^\bullet)$. Then,

$$E_2^{p,q} = \mathbb{H}^p(Y, \mathcal{E}xt^q(\tau_{>m}B^\bullet, \tau_{\leq m}B^\bullet)).$$

Moreover, the fibre of the sheaf $\mathcal{E}xt^q(\tau_{\leq m}B^\bullet, \tau_{>m}B^\bullet)$ in a point x is $[\tau_{>m}B_x^\bullet, \tau_{\leq m}B_x^\bullet[q]]$. Since $\tau_{\leq m}B_x^\bullet[q]$ is a complex of injective sheaves, $[\tau_{>m}B_x^\bullet, \tau_{\leq m}B_x^\bullet[q]]$ is equal to 0 for every $q \geq 0$. So, the sheaf $\mathcal{E}xt^q(\tau_{\leq m}B^\bullet, \tau_{>m}B^\bullet)$ is 0 for every $q \geq 0$.

Since Y has homological dimension for locally constant sheaves bounded by 1, the group $\mathbb{H}^p(Y, \mathcal{E}xt^q(\tau_{>m}B^\bullet, \tau_{\leq m}B^\bullet))$ is equal to 0 for every $p > 1$. So, $E_2^{p,q} = 0$ for every $p, q \in \mathbb{Z}$ such that $p + q = 1$.

Hence, $Ext^1(\tau_{>m}B^\bullet, \tau_{\leq m}B^\bullet) = 0$, and we conclude the proof of the existence.

Moreover, by Remark 4.6.5, the retracts of the triangle are modulated by $Ext^0(\tau_{>m}B^\bullet, \tau_{\leq m}B^\bullet)$. With the previous method, we show that, if Y has homological dimension for locally constant sheaves bounded by 0, $Ext^0(\tau_{>m}B^\bullet, \tau_{\leq m}B^\bullet) = 0$. Then, the retract of the triangle is unique and we conclude. □

Corollary 4.7.9. *If the strata X_{d-2} has the homotopy type of a 1-dimensional CW-complex of dimension bounded by 1, then here exist the intersection space complex of X for every perversity $\bar{p}$.*

Now, we illustrate the limits of our theory with a class of varieties which does not admit an intersection space complex for some perversities. With this purpose, the following proposition gives a necessary condition for the splitting of a triangle

$$\tau_{\leq m}B^\bullet \to B^\bullet \to \tau_{>m}B^\bullet \xrightarrow{[1]}$$

Proposition 4.7.10. *Let X be a topological space, let $B^\bullet$ be a bounded complex of sheaves on X and let $E_r^{p,q}$ be the local to global spectral sequence of $B^\bullet$.*

Then, if the canonical triangle

$$\tau_{\leq m}B^\bullet \to B^\bullet \to \tau_{>m}B^\bullet \xrightarrow{[1]}$$

is split, the morphisms $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ are 0 for every $r \geq 2$, $p \in \mathbb{Z}$ and $m < q \leq m + r - 1$.

Proof. Let us suppose the triangle is split and let $\lambda : B^\bullet \to \tau_{\leq m}B^\bullet$ be a retract of the canonical morphism.

Let $E_r^{p,\,q}$ be the local to global spectral sequence of $B^\bullet$, $E_r'^{p,\,q}$ the local to global spectral sequence of $\tau_{\leq m}B^\bullet$ and $\lambda_r^{p,\,q} : E_r^{p,\,q} \to E_r'^{p,\,q}$ the morphism induced by λ.

For $r = 2$,

$$\lambda_2^{p,q} : \mathbb{H}^p(X, \mathcal{H}^q(B)) \to \mathbb{H}^p(X, \mathcal{H}^q(\tau_{\leq m}B))$$

is an isomorphism if $q \leq m$ and $\mathbb{H}^p(X, \mathcal{H}^q(\tau_{\leq m}B)) = 0$ if $q > m$.

Given $r \geq 2$, suppose $\lambda_r^{p,\,q} : E_r^{p,\,q} \to E_r'^{p,\,q}$ is an isomorphism for every $q \leq m$ and $E_r'^{p,\,q} = 0$ for every $q > m$. Then, $E_{r+1}'^{p,\,q} = 0$ for every $q > m$.

Moreover, let us consider the commutative diagram:

$$\begin{array}{ccc} E_r^{p,\,q} & \xrightarrow{\lambda_r^{p,\,q}} & E_r'^{p,\,q} \\ \downarrow{\scriptstyle d_r^{p,\,q}} & & \downarrow{\scriptstyle d_r'^{p,\,q}} \\ E_r^{p+r,\,q-r+1} & \xrightarrow{\lambda_r^{p+r,\,q-r+1}} & E_r'^{p+r,\,q-r+1} \end{array}$$

If $q \leq m$, $\lambda_r^{p,\,q}$ induces an isomorphism between $\mathrm{Ker}(d_r^{p,\,q})$ and $\mathrm{Ker}(d_r'^{p,\,q})$ and $\lambda_r^{p+r,\,q-r+1}$ induces an isomorphism between $\mathrm{Im}(d_r^{p,\,q})$ and $\mathrm{Im}(d_r'^{p,\,q})$.

If $q > m$ and $q - r + 1 \leq m$, $E_r'^{p,\,q} = 0$ and, since the diagram is commutative, $d_r^{p,\,q} = d_r'^{p,\,q} = 0$. So, we deduce $d_r^{p,q}$ is 0 for every $m < q \leq m + r - 1$.

Moreover, $\mathrm{Im}(d_r^{p,\,q}) = \mathrm{Im}(d_r'^{p,\,q}) = 0$. Therefore, for every $q \leq m$, $\lambda_{r+1}^{p,\,q}$ is an isomorphism and we can finish the proof by induction. □

Corollary 4.7.11. *Let X be a topological pseudomanifold with stratification*

$$X = X_d \supset X_{d-2} \supset ... \supset X_0 \supset X_{-1} = \emptyset$$

and let k be the codimension of X_{d-2}, that is, $X_{d-2} = X_{d-k}$.

Let $\bar{p}$ a perversity and $\bar{q}$ its complementary perversity. If the local to global spectral sequence of $j_k^* i_{k*}\mathbb{Q}_{U_2}$ has any differential $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ different from 0 for some $r \geq 2$, $p \in \mathbb{Z}$ and $\bar{q}(k) < q \leq \bar{q}(k) + r - 1$, then there does not exist any intersection space complex of X with perversity $\bar{p}$.

Now, we construct an example which verify these conditions using the Hopf fibration.

Example 4.7.12. Let $\rho^\partial : S^3 \to S^2$ be the Hopf fibration and let $\rho : cyl(\rho^\partial) \to S^2$ be the cone of the fibration (see definition 4.2.3). If $s : S^2 \to cyl(\rho^\partial)$ is the vertex section, we consider the space $X := cyl(\rho^\partial)$ with the stratification

$$X \supset s(S^2)$$

Let $U := X \setminus s(S^2)$ and let $i : U \to X$ and $j : s(S^2) \to X$ be the canonical inclusions. Then, since the fibre of ρ^∂ is S^1

$$\mathcal{H}^i(j_*j^*i_*\mathbb{Q}_U) \begin{cases} \mathbb{Q}_{S^2} & \text{if}\, i = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

So, if $E_r^{p,q}$ is the hypercohomology spectral sequence of $j_*j^*i_*\mathbb{Q}_U$, $E_2^{p,q}$ is

$$q\uparrow \quad \begin{array}{c|c|c|c} 1 & \mathbb{Q} & 0 & \mathbb{Q} \\ \hline 0 & \mathbb{Q} & 0 & \mathbb{Q} \\ \hline & 0 & 1 & 2 \end{array}$$

$$\overrightarrow{p}$$

where the differential $d_2^{0,1}$ is different from 0.

Moreover, given any perversity $\bar{p}$, $\bar{p}(2) = 0$. So, applying Corollary 4.7.11, there does not exist an intersection space complex of X with stratification $X \supset s(S^2)$ with any perversity.

Hence, applying Remark 4.5.3, there does not exist any intersection space pair of X with the previous stratification.

The stratification of X given in the previous example is not natural. Since X is smooth, the natural stratification of X has no stratum different from X and $\emptyset$. The following example is more natural in the sense that the nontrivial stratum is the singular part of the variety.

Example 4.7.13. Let $\rho^\partial : S^3 \to S^2$ be the Hopf fibration and let us consider the locally trivial fibration

$$\sigma^\partial : S^3 \times_{S^2} S^3 \to S^2$$

Moreover, let $\sigma : cyl(\sigma^\partial) \to S^2$ be the cone of σ^∂ and $s : S^2 \to cyl(\sigma^\partial)$ the vertex section. Then, we define $X := cyl(\sigma^\partial)$ and we consider the stratification

$$X \supset s(S^2)$$

Let $U := X \setminus s(S^2)$ and let $i : U \to X$ and $j : s(S^2) \to X$ be the canonical inclusions. Then, since the fibre of σ^∂ is $S^1 \times S^1$

$$\mathcal{H}^i(j_*j^*i_*\mathbb{Q}_U) \begin{cases} \mathbb{Q}_{S^2} & \text{if} \quad i = 0, 2 \\ \mathbb{Q}^2_{S^2} & \text{if} \quad i = 1 \\ 0 & \text{otherwise} \end{cases}$$

So, if $E_r^{p,q}$ is the hypercohomology spectral sequence of $j_*j^*i_*\mathbb{Q}_U$, $E_2^{p,q}$ is

$$
q\uparrow \quad \begin{array}{c|c|c|c}
2 & \mathbb{Q} & 0 & \mathbb{Q} \\ \hline
1 & \mathbb{Q}^2 & 0 & \mathbb{Q}^2 \\ \hline
0 & \mathbb{Q} & 0 & \mathbb{Q} \\ \hline
 & 0 & 1 & 2
\end{array}
$$

$$\overrightarrow{p}$$

where the differentials $d_2^{0,1}$ and $d_2^{0,2}$ are different from 0.

Moreover, given any perversity $\bar{p}$, either $\bar{p}(3) = 0$ or $\bar{p}(3) = 1$. So, applying Corollary 4.7.11, there does not exist an intersection space complex of X with stratification $X \supset s(S^2)$ with any perversity.

Hence, applying Remark 4.5.3, there does not exist any intersection space pair of X with the previous stratification.

A great number of examples can be constructed with this technique. For example, if one wishes to have simply connected link and strata one, can use instead of Hopf fibration the fibration $\phi : S^7 \to S^4$ with fibre S^3.

Now, we give an example of algebraic variety for which the intersection space does not exist for the middle perversity.

Example 4.7.14. Let $Fr(2,3)$ be the frame bundle over the Grassmannian $Gr(2,3)$, that is, $Fr(2,3) := \{M \in \mathrm{Mat}(3 \times 2, \mathbb{C}) | \mathrm{rk(M)} = 2\}$ and the canonical bundle

$$\pi : Fr(2,3) \to Gr(2,3) \cong \mathbb{P}^2_{\mathbb{C}}$$

is a $GL(2,\mathbb{C})$-pincipal bundle with the action

$$
\begin{aligned}
\mathrm{GL}(2,\mathbb{C}) \times \mathrm{Fr}(2,3) &\longrightarrow Fr(2,3) \\
(A, M) &\longrightarrow A \cdot M
\end{aligned}
$$

Let $R_1^2 := \{M \in \mathrm{Mat}(2 \times 2, \mathbb{C}) | \mathrm{rk(M)} \leq 1\}$ and let us consider the action

$$
\begin{aligned}
\mathrm{GL}(2,\mathbb{C}) \times \mathrm{R}_1^2 &\longrightarrow R_1^2 \\
(A, M) &\longrightarrow A \cdot M
\end{aligned}
$$

Let $X := Fr(2,3) \times_{\mathrm{GL}(2,\mathbb{C})} R_1^2$. Since $Sing(R_1^2) = \{0\}$, we have the equality

$$Sing(X) = Fr(2,3) \times_{\mathrm{GL}(2,\mathbb{C})} \{0\} \cong Gr(2,3) \cong \mathbb{P}^2_{\mathbb{C}}$$

and the induced fibre bundle

$$
\begin{aligned}
Fr(2,n) \times_{\mathrm{GL}(2,\mathbb{C})} R_1^2 \setminus \{0\} &\longrightarrow \mathbb{P}^2_{\mathbb{C}} \\
(M_1, M_2) &\longrightarrow \pi(M_1)
\end{aligned}
$$

is the fibration of links over the singularity. The fibre of this morphism is $R_1^2 \setminus \{0\}$.

Now, let us consider the action

$$\begin{array}{rcl} \mathrm{GL}(2,\mathbb{C}) \times \mathbb{C}^2 & \longrightarrow & \mathbb{C}^2 \\ (A,(a,b)) & \longrightarrow & A \cdot \begin{pmatrix} a \\ b \end{pmatrix} \end{array}$$

and let $Y := Fr(2,3) \times_{\mathrm{GL}(2,\mathbb{C})} \mathbb{C}^2$.

The morphism

$$\begin{array}{rcl} \mathbb{C}^2 & \xrightarrow{f} & R_1^2 \\ (a,b) & \longrightarrow & \begin{pmatrix} a & a \\ b & b \end{pmatrix} \end{array}$$

is compatible with the actions. So, it induces a morphism $g : Y \to X$.

Moreover, $g^{-1}(Sing(X)) = Fr(2,3) \times_{\mathrm{GL}(2,\mathbb{C})} \{0\} \cong Gr(2,3) \cong \mathbb{P}^2_{\mathbb{C}}$ and the fibre bundle

$$\begin{array}{rcl} Fr(2,3) \times_{\mathrm{GL}(2,\mathbb{C})} \mathbb{C}^2 \setminus \{0\} & \longrightarrow & \mathbb{P}^2_{\mathbb{C}} \\ (M_1, M_2) & \longrightarrow & \pi(M_1) \end{array}$$

is the fibration of links. The fibre of this morphism is $\mathbb{C}^2 \setminus \{0\}$.

In addition,

$$\begin{tikzcd} Fr(2,3) \times_{\mathrm{GL}(2,\mathbb{C})} \mathbb{C}^2 \setminus \{0\} \arrow[rr, "g"] \arrow[dr] & & Fr(2,n) \times_{\mathrm{GL}(2,\mathbb{C})} R_1^2 \setminus \{0\} \arrow[dl] \\ & \mathbb{P}^2_{\mathbb{C}} & \end{tikzcd}$$

is a morphism of fibrations which induces in the fibre the morphism $f : \mathbb{C}^2 \setminus \{0\} \to R_1^2 \setminus \{0\}$.

Let us denote $U_X := Fr(2,n) \times_{\mathrm{GL}(2,\mathbb{C})} R_1^2 \setminus \{0\}$ and $U_Y := Fr(2,n) \times_{\mathrm{GL}(2,\mathbb{C})} \mathbb{C}^2 \setminus \{0\}$. Moreover, let $j_X : \mathbb{P}^2_C \to X$, $i_X : U_X \to X$, $j_Y : \mathbb{P}^2_C \to Y$ and $i_Y : U_Y \to Y$ be the canonical inclusions.

The morphism between fibrations g produces a morphism of complexes

$$j_{Y*} j_Y^* i_{Y*} \mathbb{Q}_{U_Y} \xrightarrow{\gamma} j_{X*} j_X^* i_{X*} \mathbb{Q}_{U_X}.$$

Moreover, $\mathbb{C}^2 \setminus \{0\}$ is homotopically equivalent to S^3, $R_1^2 \setminus \{0\}$ is homotopically equivalent to $S^3 \times S^2$ and $f : \mathbb{C}^2 \setminus \{0\} \to R_1^2 \setminus \{0\}$ induces an isomorphism between the 0-th and the third cohomology groups. Then γ induces an isomorphism between the cohomology sheaves

$$\mathcal{H}^0(j_{Y*} j_Y^* i_{Y*} \mathbb{Q}_{U_Y}) \cong \mathcal{H}^0(j_{X*} j_X^* i_{X*} \mathbb{Q}_{U_X})$$

and

$$\mathcal{H}^3(j_{Y*}j_Y^*i_{Y*}\mathbb{Q}_{U_Y}) \cong \mathcal{H}^3(j_{X*}j_X^*i_{X*}\mathbb{Q}_{U_X}).$$

Let $E_r^{p,q}$ be the local to global spectral sequence of $j_{X*}j_X^*i_{X*}\mathbb{Q}_{U_X}$, let $E_r'^{p,q}$ be the local to global spectral sequence of hypercohomology of $j_{Y*}j_Y^*i_{Y*}\mathbb{Q}_{U_Y}$, and $\gamma_r^{p,q}$: $E_r'^{p,q} \to E_r^{p,q}$ the morphism induced by γ. Then,

$$\gamma_2^{p,q} : \mathbb{H}^p(\mathbb{P}^2_C, \mathcal{H}^q(j_{Y*}j_Y^*i_{Y*}\mathbb{Q}_{U_Y})) \to \mathbb{H}^p(\mathbb{P}^2_C, \mathcal{H}^q(j_{X*}j_X^*i_{X*}\mathbb{Q}_{U_X}))$$

is an isomorphism if $q = 0, 3$.

$E_2'^{p,q}$ is

$q \uparrow$						
	3	$\mathbb{Q}$	0	$\mathbb{Q}$	0	$\mathbb{Q}$
	2	0	0	0	0	0
	1	0	0	0	0	0
	0	$\mathbb{Q}$	0	$\mathbb{Q}$	0	$\mathbb{Q}$
		0	1	2	3	4

$\overrightarrow{p}$

and it does not degenerate in $r = 2$. So, $d_4'^{0,3}$ is different from 0 since it is the unique differential different from 0 which can appear.

Moreover, $E_2^{p,q}$ is

$q \uparrow$						
	6	$\mathbb{Q}$	0	$\mathbb{Q}$	0	$\mathbb{Q}$
	5	0	0	0	0	0
	4	0	0	0	0	0
	3	$\mathbb{Q}$	0	$\mathbb{Q}$	0	$\mathbb{Q}$
	2	$\mathbb{Q}$	0	$\mathbb{Q}$	0	$\mathbb{Q}$
	1	0	0	0	0	0
	0	$\mathbb{Q}$	0	$\mathbb{Q}$	0	$\mathbb{Q}$
		0	1	2	3	4

$\overrightarrow{p}$

We prove now $d_r^{0,3} : E_r^{0,3} \to E_r^{r,3-r+1}$ is different from 0 for some $r \geq 0$. If $d_2^{0,3} = 0$, we have the following isomorphisms

$$E_4^{0,3} \cong E_3^{0,3} \cong E_2^{0,3} \cong E_2'^{0,3} \cong E_4'^{0,3}.$$

Moreover,

$$E_4^{4,0} \cong E_2^{4,0} \cong E_2'^{4,0} \cong E_4'^{4,0}$$

and the diagram

$$\begin{tikzcd} E_4^{0,3} \arrow[r, "\gamma_4^{0,3}"] \arrow[d, "d_4^{0,3}"] & E_4'^{0,3} \arrow[d, "d_4'^{0,3}"] \\ E_4^{4,0} \arrow[r, "\gamma_4^{4,0}"] & E_4'^{4,0} \end{tikzcd}$$

is cartesian.

Then, since $d_4'^{0,3}$ is not 0, $d_4^{0,3}$ is also different from 0.

If $\bar{p}$ is a perversity such that $\bar{p}(6) = 2$ and $\bar{q}$ is the complementary perversity, then we have $\bar{q}(6) = 2$. This happens for the middle perversity. Consequently, applying Corollary 4.7.11, there does not exist an intersection space complex of $X = Fr(2,3) \times_{\mathrm{GL}(2,\mathbb{C})} R_1^2$ with perversity $\bar{p}$ and stratification $X \supset Sing(X) \cong \mathbb{P}^2_C$.

Hence, applying Remark 4.5.3, there does not exist any intersection space pair of X with perversity $\bar{p}$.

4.8 Duality

In this section, we establish the duality properties of the intersection space complexes. First, we study the Verdier dual of the intersection space complex. Next, we give a version of Poincare duality for these complexes.

Let X be a topological pseudomanifold with the following stratification:

$$X = X_d \supset X_{d-2} \supset ... \supset X_0 \supset X_{-1} = \emptyset$$

Let $U_k := X \setminus X_{d-k}$ and let $i_k : U_k \to U_{k+1}$ and $j_k : X_{d-k} \setminus X_{d-k-1} \to U_{k+1}$ be the natural inclusions.

4.8.1 Verdier duality

Theorem 4.8.1. *Let $IS_{\bar{p}}$ be an intersection space complex of X with perversity $\bar{p}$ and let $\bar{q}$ be the complementary perversity of $\bar{p}$. Then, $\mathcal{D}IS_{\bar{p}}[-d]$, where $\mathcal{D}$ denotes the Verdier dual, is an intersection space complex of X with perversity $\bar{q}$.*

Proof. We have to prove $\mathcal{D}IS_{\bar{p}}[-d]$ verifies $[AXS1]_k$ for perversity $\bar{q}$ for $k = 2, ..., d$.

(a) We have a chain of isomorphisms

$$(\mathcal{D}IS_{\bar{p}}[-d])_{|U_2} \cong \mathcal{D}\mathbb{Q}_{U_2}[-d] \cong \mathbb{Q}_{U_2}.$$

(b) Let $x \in X$, the group $\mathcal{H}^i(j_x^*\mathcal{D}IS_{\bar{p}}[-d])$ is isomorphic to $\mathcal{H}^{d-i}(j_x^!IS_{\bar{p}})^v$, which is 0 if $i \notin \{0, 1, ..., d\}$.

(c_k) Let $x \in X_{d-k} \backslash X_{d-k-1}$, the group $\mathcal{H}^i(j_x^*\mathcal{D}IS_{\bar{p}}[-d])$ is isomorphic to $\mathcal{H}^{d-i}(j_x^!IS_{\bar{p}})^v$, which (by property $(d1'_k)$) is 0 if $d - i > d - \bar{p}(k) - 1$, that is, if $i \leq \bar{p}(k)$.

$(d1'_k)$ Let $x \in X_{d-k} \backslash X_{d-k-1}$, the group $\mathcal{H}^i(j_x^!\mathcal{D}IS_{\bar{p}}[-d])$ is isomorphic to $\mathcal{H}^{d-i}(j_x^*IS_{\bar{p}})^v$, which (by property (c_k)) is 0 if $d - i \leq \bar{q}(k)$, that is, if $i > d - \bar{q}(k) - 1$.

($d2'_k$) Let $x \in X_{d-k} \setminus X_{d-k-1}$, the group $\mathcal{H}^{d-\bar{q}(k)-1}(j_x^! \mathcal{D} IS_{\bar{p}}[-d])$ is isomorphic to $\mathcal{H}^{\bar{q}(k)+1}(j_x^* IS_{\bar{p}})^v$, the group $\mathcal{H}^{\bar{p}(k)+1}(j_x^* \mathcal{D} IS_{\bar{p}}[-d])$ is isomorphic to $\mathcal{H}^{d-\bar{p}(k)-1}(j_x^! IS_{\bar{p}})^v$ and the canonical morphism

$$\mathcal{H}^{d-\bar{q}(k)-1}(j_x^! \mathcal{D} IS_{\bar{p}}[-d]) \to \mathcal{H}^{\bar{p}(k)+1}(j_x^* \mathcal{D} IS_{\bar{p}}[-d])$$

is the dual morphism of $\mathcal{H}^{d-\bar{p}(k)-1}(j_x^! IS_{\bar{p}}) \to \mathcal{H}^{\bar{q}(k)+1}(j_x^* IS_{\bar{p}})$, which is the morphism 0 (by property ($d2'_k$)).

□

In Corollary 4.6.6 the space of obstructions for existence and uniqueness of intersection spaces are described. Verdier Duality $\mathcal{D}$ interchanges intersection space complexes with complementary perversities. We deduce

Corollary 4.8.2. *Let X be a topological pseudomanifold as above. Let $\bar{p}$ and $\bar{q}$ complementary perversities. An intersection space complex for perversity $\bar{p}$ exists if and only if an intersection space complex for perversity $\bar{q}$ exists. Verdier duality induces a bijection between the set of intersection space complexes for perversity $\bar{p}$ and the set of intersection space complexes for perversity $\bar{q}$.*

4.8.2 Poincare duality in the case of 2 strata

Now, suppose X has a unique non-trivial stratum. So, the stratification of X is

$$X \supset X_{d-k} \supset \emptyset$$

where k is the codimension of X_{d-k}.

According with Corollary 4.6.6, the obstruction for existence of intersection space for perversity $\bar{p}$ lives in $Ext^1(\tau_{>\bar{q}(k)} j_{k*} j_k^* i_{k*} \mathbb{Q}|_{U_k}, \tau_{\leq \bar{q}(k)} j_{k*} j_k^* i_{k*} \mathbb{Q}|_{U_k})$. Assume that the obstruction vanishes so that intersection space complexes exist. In this case the space of intersection space complexes for perversity $\bar{p}$ is parametrized by the vector space

$$E_{\bar{p}} := Hom(\tau_{>\bar{q}(k)} j_{k*} j_k^* i_{k*} \mathbb{Q}|_{U_k}, \tau_{\leq \bar{q}(k)} j_{k*} j_k^* i_{k*} \mathbb{Q}|_{U_k}).$$

Corollary 4.8.2 implies that the obstruction for existence of intersection space for perversity $\bar{q}$ vanishes and that the space

$$E_{\bar{q}} := Hom(\tau_{>\bar{p}(k)} j_{k*} j_k^* i_{k*} \mathbb{Q}|_{U_k}, \tau_{\leq \bar{p}(k)} j_{k*} j_k^* i_{k*} \mathbb{Q}|_{U_k})$$

of intersection space complexes for perversity $\bar{q}$ is isomorphic to $E_{\bar{p}}$.

Proposition 4.8.3. *Let $E_{\bar{p}}$ be the space of all intersection space complexes of X with perversity $\bar{p}$ up to isomorphisms.*

The dimensions of the vector spaces $\mathbb{H}^i(X, IS_{\bar{p}})$ with $IS_{\bar{p}} \in E_{\bar{p}}$ have a minimum and the subset

$$\{IS_{\bar{p}} \in E_{\bar{p}} | \dim(\mathbb{H}^i(X, IS_{\bar{p}})) \textit{ is minimum}\} \subset E_{\bar{p}}$$

is open for every i.

Proof. For every intersection space complex $IS_{\bar{p}} \in E_{\bar{p}}$, we have a triangle

$$IS_{\bar{p}} \to i_{k*}\mathbb{Q}_{U_2} \to \tau_{\leq \bar{q}(k)} j_{k*} j_k^* i_{k*}\mathbb{Q}_{U_2}$$

This triangle induce the long exact sequence of hypercohomology

$$... \to \mathbb{H}^i(X, IS_{\bar{p}}) \to \mathbb{H}^i(X, i_{k*}\mathbb{Q}_{U_2}) \xrightarrow{\alpha^i(IS_{\bar{p}})} \mathbb{H}^i(X, \tau_{\leq \bar{q}(k)} j_{k*} j_k^* \mathbb{Q}_{U_2}) \to ...$$

So, for every $i \in \mathbb{Z}$, there is an isomorphism

$$\mathbb{H}^i(X, IS_{\bar{p}}) \cong \mathrm{Ker}(\alpha^i(IS_{\bar{p}})) \oplus \mathrm{CoKer}(\alpha^{i-1}(IS_{\bar{p}})).$$

Moreover,

$$\dim(\mathrm{CoKer}(\alpha^i(IS_{\bar{p}}))) = \dim(\mathbb{H}^i(X, \tau_{\leq \bar{q}(k)} j_{k*} j_k^* \mathbb{Q}_{U_2})) - \dim(\mathbb{H}^i(X, i_{k*}\mathbb{Q}_{U_2})) +$$

$$+ \dim(\mathrm{Ker}(\alpha^i(IS_{\bar{p}})))$$

So, $\dim(\mathbb{H}^i(X, IS_{\bar{p}}))$ is minimum if and only if $\dim(\mathrm{Ker}(\alpha^i(IS_{\bar{p}})))$ and $\dim(\mathrm{Ker}(\alpha^{i-1}(IS_{\bar{p}})))$ are minimum.

The morphism $\alpha^i(IS_{\bar{p}})$ is the morphisms induced in hypercohomology by the composition

$$i_{k*}\mathbb{Q}_{U_2} \xrightarrow{a} j_{k*} j_k^* i_{k*}\mathbb{Q}_{U_2} \xrightarrow{\lambda(IS_{\bar{p}})} \tau_{\leq \bar{q}(k)} j_{k*} j_k^* i_{k*}\mathbb{Q}_{U_2}$$

where a is the canonical morphisms and $\lambda(IS_{\bar{p}})$ is a retraction of the natural truncation morphism $f : \tau_{\leq \bar{q}(k)} j_{k*} j_k^* i_{k*}\mathbb{Q}_{U_2} \to j_{k*} j_k^* i_{k*}\mathbb{Q}_{U_2}$.

Let us denote by

$$a^i : \mathbb{H}^i(X, i_{k*}\mathbb{Q}_{U_2}) \to \mathbb{H}^i(X, j_{k*} j_k^* i_{k*}\mathbb{Q}_{U_2}),$$

$$\lambda^i(IS_{\bar{p}}) : \mathbb{H}^i(X, j_{k*} j_k^* i_{k*}\mathbb{Q}_{U_2}) \to \mathbb{H}^i(X, \tau_{\leq \bar{q}(k)} j_{k*} j_k^* i_{k*}\mathbb{Q}_{U_2}),$$

$$f^i : \mathbb{H}^i(X, \tau_{\leq \bar{q}(k)} j_{k*} j_k^* i_{k*}\mathbb{Q}_{U_2}) \to \mathbb{H}^i(X, j_{k*} j_k^* i_{k*}\mathbb{Q}_{U_2})$$

the morphisms induced in hypercohomology. Then, we have

$$\dim(\mathrm{Ker}(\alpha^i(IS_{\bar{p}}))) = \dim((a^i)^{-1}(\mathrm{Ker}(\lambda^i(IS_{\bar{p}})))) = \dim(\mathrm{Ker}(a^i)) +$$

$$+ \dim(\mathrm{Im}(a^i) \cap \mathrm{Ker}(\lambda^i(IS_{\bar{p}})))$$

Hence, $\dim(\mathrm{Ker}(\alpha^i(IS_{\bar{p}})))$ is minimum if and only if $\dim(\mathrm{Im}(a^i) \cap \mathrm{Ker}(\lambda^i(IS_{\bar{p}})))$ is minimum.

Since $\lambda^i(IS_{\bar{p}}) \circ f^i$ is the identity, the homomorphism $\lambda^i(IS_{\bar{p}})$ is surjective and we have the equality

$$\dim(\mathrm{Ker}(\lambda^i(IS_{\bar{p}}))) = \dim(\mathbb{H}^i(X, j_{k*} j_k^* i_{k*}\mathbb{Q}_{U_2})) - \dim(\mathbb{H}^i(X, \tau_{\leq \bar{q}(k)} j_{k*} j_k^* i_{k*}\mathbb{Q}_{U_2})).$$

So, $\dim(\mathrm{Ker}(\lambda^i(IS_{\bar{p}})))$ is independent of $IS_{\bar{p}}$. Then, for every $IS_{\bar{p}} \in E_{\bar{p}}$, there is an isomorphism

$$\mathbb{H}^i(X, j_{k*}j_k^*i_{k*}\mathbb{Q}_{U_2}))/\mathrm{Ker}(\lambda^i(IS_{\bar{p}})) \cong \mathbb{Q}^{d^i}$$

where $d^i := \dim(\mathbb{H}^i(X, \tau_{\leq \bar{q}(k)} j_{k*}j_k^*i_{k*}\mathbb{Q}_{U_2}))$.

Now, consider the composition of morphisms

$$\begin{tikzcd}
\mathbb{H}^i(X, i_{k*}\mathbb{Q}_{U_2}) \arrow[r, "a^i"] \arrow[dr, "\phi(IS_{\bar{p}})"'] & \mathbb{H}^i(X, j_{k*}j_k^*i_{k*}\mathbb{Q}_{U_2}) \arrow[d, "\pi(IS_{\bar{p}})"] \\
& \mathbb{H}^i(X, j_{k*}j_k^*i_{k*}\mathbb{Q}_{U_2}))/\mathrm{Ker}(\lambda^i(IS_{\bar{p}})) \cong \mathbb{Q}^{d^i}
\end{tikzcd}$$

where $\pi(IS_{\bar{p}})$ is the canonical projection.

Then, $\mathrm{Im}(a^i) \cap \mathrm{Ker}(\lambda^i(IS_{\bar{p}}))$ gets the minimum dimension when the morphism $\phi(IS_{\bar{p}})$ gets the maximum rank, which happens in an open subset. □

Definition 4.8.4. The *general i-th Betti number* of the intersection space complexes of X with perversity $\bar{p}$ is the minimum of the dimensions of the vector spaces $\mathbb{H}^i(X, IS_{\bar{p}})$ with $IS_{\bar{p}} \in E_{\bar{p}}$.

Definition 4.8.5. A *general intersection space complex* of X with perversity $\bar{p}$ is an intersection space complex $IS_{\bar{p}} \in E_{\bar{p}}$ such that $\dim(\mathbb{H}^i(X, IS_{\bar{p}}))$ is the general i-th Betti number for $i = 0, 1, ..., d$.

Theorem 4.8.6. *Let $\bar{p}$ be a perversity and let $\bar{q}$ be its complementary perversity. If $IS_{\bar{p}}$ is a general intersection space complex of X with perversity $\bar{p}$ and $IS_{\bar{q}}$ is a general intersection space complex of X with perversity $\bar{q}$, then, for $i = 0, 1, ..., d$, there is an isomorphism of $\mathbb{Q}$-vector spaces*

$$\mathbb{H}^i(X, IS_{\bar{p}}) \cong \mathbb{H}^{d-i}(X, IS_{\bar{q}})^v$$

Proof. Given any intersection space complex $IS_{\bar{p}}$ of X with perversity $\bar{p}$, we have

$$\mathbb{H}^i(X, \mathcal{D}IS_{\bar{p}}[-d])^v \cong \mathbb{H}^{i-d}(X, \mathcal{D}IS_{\bar{p}})^v \cong \mathbb{H}^{d-i}(X, IS_{\bar{p}}).$$

Applying Theorem 4.8.1, the complex $\mathcal{D}IS_{\bar{p}}[-d]$ is an intersection space complex of X with perversity $\bar{q}$. We denote $IS_{\bar{q}} := \mathcal{D}IS_{\bar{p}}[-d]$.

Suppose $IS_{\bar{q}}$ is not a general intersection space complex of X. Then, there exist another intersection space complex of X with perversity $\bar{q}$, $IS'_{\bar{q}}$, such that we have the strict inequality

$$\sum_{i=0}^{d} \dim(\mathbb{H}^i(X, IS'_{\bar{q}})) < \sum_{i=0}^{d} \dim(\mathbb{H}^i(X, IS_{\bar{q}})).$$

Consequently we have,

$$\sum_{i=0}^{d} \dim(\mathbb{H}^i(X, \mathcal{D}IS'_{\bar{q}}[-d])) = \sum_{i=0}^{d} \dim(\mathbb{H}^i(X, IS'_{\bar{q}})^v) < \sum_{i=0}^{d} \dim(\mathbb{H}^i(X, IS_{\bar{q}})^v) =$$

$$= \sum_{i=0}^{d} \dim(\mathbb{H}^i(X, \mathcal{D}IS_{\bar{q}}[-d])) = \sum_{i=0}^{d} \dim(\mathbb{H}^i(X, IS_{\bar{p}})).$$

So, $IS_{\bar{p}}$ is not a general intersection space complex of X.

We deduce that if $IS_{\bar{p}}$ is a general intersection space complex, then $IS_{\bar{q}}$ is also a general intersection space complex. So, there are isomorphisms

$$\mathbb{H}^i(X, IS_{\bar{p}}) \cong \mathbb{H}^{d-i}(X, IS_{\bar{q}})^v$$

for some general intersection space complexes $IS_{\bar{p}}$ and $IS_{\bar{q}}$.

Since the hypercohomology groups of general intersection space complexes with the same perversity are isomorphic, we conclude. □

Bibliography

[AL17] M. Agustín and K. Langlois. On intersection cohomology with torus actions of complexity one. Rev. Mat. Complut. 31 (2018), no. 1, 163–186.

[AH06] K. Altmann, J. Hausen. *Polyhedral divisors and algebraic torus actions.* Math. Ann. 334 (2006), no. 3, 557–607.

[AHS08] K. Altmann, J. Hausen, H. Suess. *Gluing affine torus actions via divisorial fans.* Transform. Groups 13 (2008), no. 2, 215–242.

[Ban07] M. Banagl. *Topological invariants of stratified spaces.* Springer, Berlin, 2007.

[Ban09] M. Banagl. *Singular Spaces and Generalized Poincaré Complexes*, Electron. Res. Announc. Math. Sci. **16** (2009), 63-73.

[Ban10] M. Banagl. *Intersection spaces, spatial homology truncation and string theory* Lecture notes in mathematics **1997** Springer-Verlag, Berlin, Heilderberg, 2010.

[Ban12] M. Banagl. *First Cases of Intersection Spaces in Stratification Depth 2*, J. of Singularities **5** (2012), 57 - 84.

[BBM] M. Banagl. N. Budur, L. Maxim. *Intersection Spaces, Perverse Sheaves and Type IIB String Theory (pdf)*, Adv. Theor. Math. Phys. **18** (2014), no. 2, pp. 363 - 399.

[BaCh] M. Banagl, B. Chriestenson. *Intersection Spaces, Equivariant Moore Approximation and the Signature*, J. of Singularities **16** (2017), pp. 141 - 179.

[BaHu] M. Banagl, E. Hunsicker. *Hodge Theory for Intersection Space Cohomology*, arXiv:1502.03960 (2015).

[BM1] M. Banagl. L. Maxim. *Intersection Spaces and Hypersurface Singularities*, J. of Singularities **5** (2012), 48 - 56.

[BM2] M. Banagl. L. Maxim. *Deformation of Singularities and the Homology of Intersection Spaces*, J. Topol. Anal. **4** (2012), no. 4, 413 - 448.

[BBKK99] G. Barthel, J-P. Brasselet, F. Karl-Heinz, L. Kaup. *Equivariant intersection cohomology of toric varieties.* Algebraic geometry: Hirzebruch 70 (Warsaw, 1998), 45–68, Contemp. Math., 241, Amer. Math. Soc., Providence, RI, 1999.

[BB96] V. Batyrev, L. Borisov. *Mirror duality and string-theoretic Hodge numbers.* Invent. Math. 126 (1996), no. 1, 183–203.

[BBD82] A. Beilison, J. Bernstein, P. Deligne. *Faisceaux pervers.* Analysis and topology on singular spaces, I (Luminy, 1981), 5–171, Astérisque, 100, Soc. Math. France, Paris, 1982.

[Bri99] M. Brion. *Rational smoothness and fixed points of the torus.* Dedicated to the memory of Claude Chevalley. Transform. Groups 4 (1999), No 2-3, 127–156.

[CMS08] S. Cappell, L. Maxim, J. Shaneson. *Hodge genera of algebraic varieties. I.* Comm. Pure Appl. Math. 61 (2008), no. 3, 422–449. Bull. Amer. Math. Soc. (N.S.) 46 (2009), no.4, 535–633.

[CaMi] M. A. de Cataldo, L. Migliorini. *The Decomposition Theorem, perverse sheaves and the topology of algebraic maps* Bull. Amer. Math. Soc. **46**, no 4, October 2009, 535-633.

[CMM15] M. A. de Cataldo, L. Migliorini, M. Mustata. *The combinatorics and topology of proper toric maps.* J. Reine Angew. Math. DOI: 10.1515/crelle-2015-0104.

[CLS11] D. A. Cox. J. Little, H. Schenck. *Toric varieties.* Graduate Studies in Mathematics, 124. American Mathematical Society, Providence, RI, 2011.

[DL91] J. Denef, F. Loeser. *Weights of exponential sums, intersection cohomology, and Newton polyhedra.* Ivent. Math. 106 (1991), no.2, 275–294.

[Dim04] A. Dimca. *Sheaves in topology.* Universitext. Springer-Verlag, Berlin, 2004.

[FK86] K.-H. Fieseler, L. Kaup. *Intersection Homology of $\mathbb{C}^*$-Surfaces.* Aspects of mathematics and its applications, 439–463, North-Holland Math. Library, 34, North-Holland, Amsterdam, 1986.

[Fie91] K.-H. Fieseler. *Rational intersection cohomology of projective toric varieties.* J. Reine Angew. Math. 413 (1991), 88–98.

[FZ03] H. Flenner, M. Zaidenberg. *Normal affine surfaces with $\mathbb{C}^\star$-actions.* Osaka J. Math. 40 (2003), no. 4, 981–1009.

[Ful93] W. Fulton. *Introduction to toric varieties.* Annals of Mathematics Studies, 131. The William H. Roever Lectures in Geometry. Princeton University Press, Princeton, NJ, 1993.

[Ge] C. Geske. *Algebraic intersection spaces*, arXiv:1802.03871v1 (2018).

[GWPL] C. G. Gibson, K. Wirthmller, A. A. du Plessis, E. J. N. Looijenga. *Topological stability of smooth mappings*, Lecture Notes in Math., Vol. 552, Springer-Verlag, Berlin, New York, 1976.

[GM83] M. Goresky, R. MacPherson. *Intersection homology theory II.* Invent. Math. 72 (1983), no.1, 77–129.

[HTT08] R. Hotta, K. Takeuchi, T. Tanisaki. *D-modules, perverse sheaves, and representation theory.* Progress in Mathematics, 236. Birkhauser Boston, Inc. Boston, MA, 2008.

[KKMS73] G. Kempf, F. Knudsen, D. Mumford, B. Saint-Donat. *Toroidal embeddings.* I. Lecture Notes in Mathematics, Vol. 339. Springer-Verlag, Berlin-New York, 1973.

[Kir88] F. Kirwan. *Intersection homology and torus actions.* J. Amer. Math. Soc. 1, no.2, (1988).

[Kl] M. Klimczak. *Poincaré duality for spaces with isolated singularities.* arXiv:150707407v3 (2016).

[LLM16] A. Laface, A. Liendo, J. Moraga. *Topology of T-Varieties.* arXiv:1503.06023

[Lan14] K. Langlois. *Polyhedral divisors and torus actions of complexity one over arbitrary fields.* J. Pure Appl. Algebra (2014), no 6, 2015–2045.

[LT16] K. Langlois, R. Terpereau. *On the geometry of normal horospherical G-varieties of complexity one.* J. Lie Theory 26 (2016), no.1, 049–078.

[LPR16] K. Langlois, C. Pech, M. Raibaut. *Stringy invariants for horospherical varieties of complexity one.* arXiv:1511.03852

[Lun73] D. Luna. *Slice étales.* Sur les groupes algébriques, pp. 81–105. Bull. Soc. Math. France, Paris, Memoire 33 Soc. Math. France, Paris. 1973.

[Mas] W. S. Massey. *A basic course in algebraic topology* Graduate text in mathematics **127** Springer-Verlag, New York, 1991.

[Max] L. Maxim. *Intersection spaces, perverse sheaves and string theory.* Journal of Singularities **15** (2016), 118-125

[PS11] L. Petersen, H. Suess. *Torus invariant divisors.* Israel J. Math. 182 (2011), 481–504.

[Ram] S. Ramanan. *Global calculus* Graduate Studies in Mathematics, 65. American Mathematical Society, Porvidence, Rhode Island, 2004.

[Ros56] M. Rosenlicht. *Some basic theorems on algebraic groups.* Amer. J. Math. **78** (1956), 401–443.

[Ros63] M. Rosenlicht. *A remark on quotient spaces.* An. Acad. Brasil. Ci. 35 1963 487–489.

[Sum74] H. Sumihiro. *Equivariant completion.* J. Math. Kyoto Univ. 14 (1974), 1–28.

[Sta87] R. Stanley. *Generalized H-vector, intersection cohomology of toric varieties, and related results.* Commutative algebra and combinatorics (Kyoto, 1985), 187–213, Adv. Stud. Pure Math., 11, North-Holland, Amsterdam, 1987.

[Tim00] D. A. Timashëv. *Cartier divisors and geometry of normal G-varieties.* Transform. Groups 5 (2000), no. 2, 181–204.

[Tim08] D.A. Timashëv. *Torus actions of complexity one.* Toric topology, 349–364, Contemp. Math., 460, Amer. Math. Soc., Providence, RI, 2008.

[Tim11] D. A. Timashëv. *Homogeneous spaces and equivariant embeddings.* Encyclopaedia of Mathematical Sciences, 138. Invariant Theory and Algebraic Transformation Groups, 8. Springer, Heidelberg, 2011.

If we knew what it was we were doing,
it would not be called research, would it?
Albert Einstein